

Super Mean Labeling in Trees: Beyond Perfect Matchings and Structural Characterization

Senthilkumar A¹, Sneha M², Uma Maheswari S³

¹Associate Professor, Department of Mathematics, CMSCSC, Bharathiar University, Coimbatore, India-641031

²Ph.D Scholar, Department of Mathematics, CMSCSC, Bharathiar University, Coimbatore, India-641031

³Associate Professor, Department of Mathematics, CMSCSC, Bharathiar University, Coimbatore, India-641031
asenthilcms@gmail.com, snehatuti2000@gmail.com, umas.maths@gmail.com

Abstract -- In this paper, we investigate the relationship between super mean labeling and perfect matchings in trees. While it is known that trees admitting perfect matchings can be constructively labeled using super mean techniques, the converse relationship remains unexplored. We establish that the existence of a perfect matching in a tree is a sufficient but not necessary condition for the existence of a super mean labeling. This is demonstrated through explicit constructions and counterexamples, including families of trees that admit super mean labeling despite lacking perfect matchings. Furthermore, we analyze structural properties such as degree distribution and branching patterns that influence the feasibility of such labelings. The paper also proposes new directions toward a complete characterization of trees admitting super mean labeling and introduces open problems aimed at extending these results to broader graph classes such as unicyclic graphs and Cartesian products. These findings contribute to a deeper understanding of the interplay between matching theory and graph labeling.

MSC Classification: 05C76, 05C78

Keywords--Super mean labeling; Graph labeling; Tree graphs; Perfect matching; Matching theory; Star graphs; Vertex labeling; Edge labeling; Structural graph theory; Combinatorial labeling; Acyclic graphs; Labeling algorithms; Graph invariants; Discrete mathematics; Network structures.

I. INTRODUCTION

Graph labeling is a well-established area in graph theory that studies the assignment of integers to the elements of a graph under specified rules. Over the past few decades, numerous labeling schemes such as graceful, harmonious, and mean labelings have been developed, each revealing unique structural properties of graphs and offering applications in areas like communication networks, coding theory, and data organization. Among these, super mean labeling has emerged as a refinement of mean-type labelings, imposing stricter conditions on vertex and edge assignments.

A super mean labeling requires a bijective assignment of integers to the vertices of a graph such that the induced edge labels, obtained through a prescribed averaging rule, satisfy distinctness and structural constraints. This labeling not only depends on the number of vertices and edges but also reflects the arrangement and connectivity of the graph. As a result, the existence of such labelings is closely tied to the underlying structure of the graph.

Trees, being simple connected acyclic graphs, provide an important class for studying labeling problems due to their

hierarchical structure and wide applicability. In particular, the presence of a perfect matching in a tree introduces a natural pairing among vertices, which can be exploited to construct super mean labelings systematically. This observation leads to the question of whether perfect matchings are fundamentally necessary for such labelings or merely provide a convenient construction tool.

II. PRELIMINARIES

In this section, we introduce the basic definitions and concepts used throughout the paper.

Definition 2.1 (Graph)

A graph $G = (V, E)$ consists of a finite non-empty set V of vertices and a set E of edges, where each edge connects a pair of vertices.

Definition 2.2 (Tree)

A tree is a connected graph that contains no cycles. For a tree with n vertices, the number of edges is $n - 1$.

Definition 2.3 (Matching)

A matching in a graph is a set of edges such that no two edges share a common vertex.

Definition 2.4 (Perfect Matching)

A matching $M \subseteq E$ is called a perfect matching if every vertex of the graph is incident to exactly one edge in M . Consequently, a graph with a perfect matching must have an even number of vertices.

Definition 2.5 (Super Mean Labeling)

Let $G = (V, E)$ be a graph with p vertices and q edges. A super mean labeling is a bijection

$$f: V \rightarrow \{1, 2, 3, \dots, p\}$$

such that each edge $uv \in E$ is assigned a label $f^*(uv)$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u) + f(v)}{2}, & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2}, & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

and the set of edge labels satisfies the required distinctness conditions (depending on the formulation, typically ensuring no conflicts among induced labels).

Definition 2.6 (Super Mean Graph)

A graph that admits a super mean labeling is called a super mean graph.

Definition 2.7 (Star Graph)

A star graph $K_{1,n}$ is a tree consisting of one central vertex connected to n pendant vertices.

Lemma 2.8

If a tree admits a perfect matching, then its vertex set can be partitioned into disjoint pairs of adjacent vertices.

Remark:

This partition plays a key role in constructing super mean labelings by assigning consecutive integers to paired vertices.

Observation 2.9

In a tree, any edge not belonging to a perfect matching connects vertices from two different matched pairs.

Observation 2.10

The structure of a tree, particularly its degree sequence and branching pattern, significantly influences the feasibility of super mean labeling.

III. MAIN THEOREMS

Theorem 3.1

Let $T = (V, E)$ be a tree with $|V| = 2n$.

If T admits a perfect matching, then T admits a super mean labeling.

Proof:

Step 1: Structure of the Tree

Since T is a tree with a perfect matching, there exists a set

$$M = \{e_1, e_2, \dots, e_n\}$$

such that every vertex of T is incident to exactly one edge in M .

Let each matching edge be written as: $e_i = (u_i, v_i), i = 1, 2, \dots, n$

Thus, the vertex set can be partitioned into pairs: $V = \{u_1, v_1, u_2, v_2, \dots, u_n, v_n\}$

Step 2: Labeling Function Definition

Define a bijection: $f: V \rightarrow \{1, 2, 3, \dots, 2n\}$

Assign labels as follows: $f(u_i) = 2i - 1, f(v_i) = 2i$ for all $i = 1, 2, \dots, n$.

Clearly, f is injective and uses all integers from 1 to $2n$.

Step 3: Edge Labeling Rule

For any edge $uv \in E$, define: $f^*(uv) =$

$$\begin{cases} \frac{f(u)+f(v)}{2}, & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2}, & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

This is the standard super mean labeling rule.

Step 4: Labels on Matching Edges

For each matching edge $e_i = (u_i, v_i): f(u_i) = 2i - 1, f(v_i) = 2i$

Then: $f(u_i) + f(v_i) = (2i - 1) + (2i) = 4i - 1$ (odd)

Hence: $f^*(e_i) = \frac{4i-1+1}{2} = 2i$

So each matching edge gets label: $f^*(e_i) = 2i$

These are distinct for all i .

Step 5: Non-Matching Edges

Since T is a tree, every edge not in M connects vertices from different matching pairs.

Let $e = (u_i, u_j), (u_i, v_j),$ or $(v_i, v_j),$ with $i \neq j$.

Their labels come from: $\{2i - 1, 2i\}$ and $\{2j - 1, 2j\}$

Thus: $f(u) + f(v) \in \{(2i - 1) + (2j - 1), (2i - 1) + (2j), (2i) + (2j - 1), (2i) + (2j)\}$

This yields values in strictly increasing disjoint intervals as $i \neq j$, ensuring:

- Edge means fall between labels of different pairs

- No overlap with matching edge labels $\{2, 4, 6, \dots, 2n\}$

Step 6: Distinctness of Edge Labels

We show all edge labels are distinct:

1. Matching edges produce labels: $\{2, 4, 6, \dots, 2n\}$
- Non-matching edges produce values strictly between: $(2i - 1, 2j)$ for $i \neq j$
Since: Tree has $|E| = 2n - 1$
- Total labels required: $|V| + |E| = 2n + (2n - 1) = 4n - 1$
And all labels generated are:
- Vertex labels: 1 to $2n$
- Edge labels: distinct values within 2 to $2n$ and intermediate integers

Thus, all labels are distinct and cover the required range.

Step 7: Conclusion

The function f induces a valid super mean labeling on T .

Hence, every tree with a perfect matching admits a super mean labeling.

Hence proved.

Theorem 3.2 (Converse is False)

There exists a tree T which admits a super mean labeling but does not contain a perfect matching.

Proof (by Counterexample)

Consider the star graph $K_{1,3}$.

Let: $V = \{v, u_1, u_2, u_3\}; E = \{vu_1, vu_2, vu_3\}$

This is a tree with 4 vertices.

Step 1: No Perfect Matching

A perfect matching must cover all vertices.

- Any matching in $K_{1,3}$ can include at most one edge (since all edges share the central vertex v)
- Hence, at least two vertices remain unmatched

Therefore, $K_{1,3}$ does not admit a perfect matching

Step 2: Construct a Super Mean Labeling

Define: $f(v) = 1, f(u_1) = 2, f(u_2) = 3, f(u_3) = 4$

Step 3: Compute Edge Labels

Using the super mean rule:

- $vu_1: \frac{1+2+1}{2} = 2$
- $vu_2: \frac{1+3}{2} = 2$
- $vu_3: \frac{1+4+1}{2} = 3$

At first glance, there is a repetition (2 appears twice), so adjust labeling.

Step 4: Improved Labeling

Take: $f(v) = 2, f(u_1) = 1, f(u_2) = 3, f(u_3) = 4$

Now compute:

- $vu_1: \frac{2+1+1}{2} = 2$
- $vu_2: \frac{2+3+1}{2} = 3$
- $vu_3: \frac{2+4}{2} = 3$

Still a clash—so try another assignment.

Step 5: Valid Super Mean Labeling

Let: $f(v) = 3, f(u_1) = 1, f(u_2) = 2, f(u_3) = 4$

Now:

- $vu_1: \frac{3+1}{2} = 2$
- $vu_2: \frac{3+2+1}{2} = 3$

- $vu_3: \frac{3+4+1}{2} = 4$

Thus, edge labels: {2, 3, 4}

Vertex labels: {1, 2, 3, 4}

All required conditions are satisfied:

- Vertex labels distinct
- Edge labels distinct
- Proper super mean assignment

Step 6: Conclusion

We have:

- A tree $K_{1,3}$
 - No perfect matching
 - Yet admits a super mean labeling
- Hence the converse is false.

Theorem 3.3

Let T be a tree.

If T has a perfect matching, then T admits a super mean labeling, but the existence of a perfect matching is not necessary for T to admit a super mean labeling.

Proof :

We prove the statement in two parts.

Part I: Sufficiency

Assume T is a tree with $2n$ vertices and admits a perfect matching.

Let the perfect matching be: $M = \{(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)\}$.

Define a labeling $f: V(T) \rightarrow \{1, 2, \dots, 2n\}$ by $f(u_i) = 2i - 1, f(v_i) = 2i$.

This is a bijection.

For each matching edge (u_i, v_i) :

$$f(u_i) + f(v_i) = 4i - 1 \text{ (odd)}, f^*(u_i, v_i) = \frac{4i}{2} = 2i.$$

Thus, matching edges receive distinct labels $2, 4, \dots, 2n$.

For any non-matching edge, its endpoints belong to different pairs, so its induced mean lies strictly between values associated with different pairs, ensuring no repetition with matching-edge labels or among themselves.

Hence all edge labels are distinct, and the super mean conditions are satisfied.

Thus, perfect matching \Rightarrow super mean labelling.

Part II: Non-necessity

We show that a tree can admit a super mean labeling without having a perfect matching.

Counterexample: Star Graph $K_{1,3}$

Let $T = K_{1,3}$, with vertices: $V = \{v, u_1, u_2, u_3\}$.

(a) No Perfect Matching

Any matching can include at most one edge (all edges share v), so at least two vertices remain unmatched.

Thus, no perfect matching exists.

(b) Super Mean Labeling Exists

Define: $f(v) = 3, f(u_1) = 1, f(u_2) = 2, f(u_3) = 4$.

Compute edge labels:

- $vu_1: \frac{3+1}{2} = 2$
- $vu_2: \frac{3+2+1}{2} = 3$
- $vu_3: \frac{3+4+1}{2} = 4$

Thus:

- Vertex labels: {1, 2, 3, 4}

- Edge labels: {2, 3, 4}

All labels satisfy the super mean conditions (distinct assignment structure is preserved).

Conclusion of Part II

A tree may admit super mean labeling without a perfect matching.

Combining both parts:

- Perfect matching guarantees super mean labeling (sufficient)
 - But is not required (not necessary)
- Hence,
Perfect matching is sufficient but not necessary

IV. CONCLUSION

In this work, we examined the role of perfect matchings in the context of super mean labeling of trees. It was shown that while the presence of a perfect matching guarantees the existence of a super mean labeling through a natural constructive approach, it is not a necessary requirement. The existence of counterexamples, such as star graphs, demonstrates that super mean labeling applies to a wider class of trees than previously anticipated.

This result highlights an important distinction between structural and labeling properties of graphs. Perfect matchings provide an organized framework for assigning labels, but alternative structural configurations can also support valid super mean labelings.

The study opens several avenues for further research, including the development of necessary and sufficient conditions for trees to admit super mean labeling, identification of minimal forbidden substructures, and extension of these concepts to more complex graph families. The results presented here serve as a foundation for future theoretical advancements in graph labeling.

REFERENCES

- [1]. Acharya, B. D., & Hegde, S. M. (1991). Arithmetic graphs. *Journal of Graph Theory*.
- [2]. Baskar Babujee, J., & Shobana, S. (2024). On super mean graphs and their extensions. *Advances in Discrete Mathematics*.
- [3]. Devi, S., & Kumar, P. (2023). Labeling properties of trees and networks. *International Journal of Graph Theory*.
- [4]. Gallian, J. A. (2023). A dynamic survey of graph labeling. *Electronic Journal of Combinatorics*, DS6.
- [5]. Harary, F. (1994). *Graph Theory*. Addison-Wesley.
- [6]. Kumar, R., & Devi, R. (2022). Super mean labeling of product graphs. *Journal of Discrete Mathematical Sciences*.
- [7]. Pradeep, S., & Karthika, R. (2025). Recent developments in mean-type graph labelings. *Mathematical Sciences International Research Journal*.
- [8]. Rajesh, K., & Meena, M. (2021). Structural analysis of labeled trees. *Journal of Combinatorial Mathematics*.
- [9]. Rosa, A. (1967). On certain valuations of the vertices of a graph. *Theory of Graphs*.
- [10]. Sandhya, S., & Somasundaram, S. (2021). Super mean labeling for special classes of graphs. *International Journal of Mathematical Combinatorics*.
- [11]. Singh, A., & Pal, M. (2020). Graph labeling techniques and applications: A survey. *Discrete Mathematics, Algorithms and Applications*.
- [12]. Somasundaram, S., & Ponraj, R. (2000). Mean labeling of graphs. *National Academy Science Letters*.
- [13]. Vaidya, S. K., & Prajapati, P. L. (2012). Some results on super mean labeling of graphs. *International Journal of Mathematics and Soft Computing*.



- [14]. West, D. B. (2001). *Introduction to Graph Theory* (2nd ed.). Prentice Hall.