

The Unified Polynomial Set $Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda)$

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Abstract— In this paper, a polynomial set $Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda)$ is introduced which yields the above in mentioned polynomial as its particular cases. When the limit technique is applied. After defining the polynomial by R.K. Singh [6,7,8], the series representation elementary properties of Hypergeometric form and recurrence relations have been obtained. Moreover, one contiguous relation has also been deduced the paper is concluded by giving some interesting results and then deducing Hypergeometric form for $Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda)$.

Keywords— Different recurrence relations, series representation.

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1. Definition: We define the sequence $\{Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda)\}_{n=0}^{\infty}$ by means of the generating relation.

$$\sum_{n=0}^{\infty} Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda) \frac{t^n}{[n]} = \frac{c \left(\frac{t}{c}\right)^\lambda (1-nt)^{-\alpha}}{(1-at)^{-\beta} - b} \quad (1.1)$$

where α, β, a, b, c are real numbers and λ is non-negative integer.

2. Series representation: By definition

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda) \frac{t^n}{[n]} &= \frac{c \left(\frac{t}{c}\right)^\lambda (1-xt)^{-\alpha}}{(1-at)^{-\beta} - b} \\ &= \frac{c \left(\frac{t}{c}\right)^\lambda}{(-a)} \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n t^n}{[n]} \left\{ 1 - \frac{(1-at)^{-\beta}}{b} \right\}^{-1} \\ &= \frac{c^{1-\lambda}}{(-b)} \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n t^{n+\lambda}}{[n]} \sum_{m=0}^{\infty} \frac{(1-at)^{-\beta m}}{b^m} \\ &= (-c)^{1-\lambda} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_n x^n t^{n+\lambda}}{[n] b^{m+1}} \sum_{q=0}^{\infty} \frac{(\beta m)_q}{q} a^q t^q \\ &= (-c)^{1-\lambda} \sum_{n,m=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\alpha)_n (\beta m)_s x^n a^q t^{n+\lambda+q}}{[n] [q] b^{m+1}} \\ &= (-c)^{1-\lambda} \sum_{n,m=0}^{\infty} \sum_{q=0}^n \frac{(\alpha)_{n-q} x^{n-q} a^q t^{n+\lambda} (\beta m)_q}{[n-q] [q] b^{m+1}} \quad (2.1) \\ &= (-c)^{1-\lambda} \sum_{n,m=0}^{\infty} \frac{(\alpha)_n x^n t^{n+\lambda} (\beta m)_q}{[n] b^{m+1}} \sum_{q=0}^n \frac{(-n)_q (\beta m)_q}{(1-\alpha-n)_q [q] x^q} \end{aligned}$$

$$\begin{aligned} &= (-c)^{1-\lambda} \sum_{n,m} \frac{(\alpha)_n x^n t^{n+\lambda}}{[n] b^{m+1}} {}_2F_1 \left[\begin{matrix} -n, \beta m; \\ 1-\alpha-n \end{matrix} ; \frac{a}{x} \right] \\ &= (-c)^{1-\lambda} \sum_{n=0}^{\infty} \frac{(\alpha)_{n-\lambda} x^{n-\lambda} t^n}{[n-\lambda]} \sum_{m=0}^{\infty} b^{-m-1} {}_2F_1 \left[\begin{matrix} -n+\lambda, \beta m; \\ 1-\alpha-n+\lambda \end{matrix} ; \frac{a}{x} \right] \end{aligned}$$

Equating the coefficients of t^n on both sides, we have

$$\beta_n^{(\alpha, \beta)}(x; a, b, c, \lambda) = - \frac{(\alpha)_{n-\lambda} x^{n-\lambda} (1-\lambda) [n]}{[n-\lambda]} \sum_{m=0}^{\infty} b^{-m-1} {}_2F_1 \left[\begin{matrix} -n+\lambda, \beta m; \\ 1-\alpha-n+\lambda \end{matrix} ; \frac{a}{x} \right] \quad (2.2)$$

The relation (2.2) may put in the form

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda) &= \frac{-c^{1-\lambda}}{b} \sum_{n=0}^{\infty} \sum_{q=0}^{n-\lambda} \frac{(\alpha)_{n-q-\lambda} x^{n-q-\lambda} a^q}{[n-q-\lambda] [q]} \sum_{m=0}^{\infty} \frac{(\beta m)_q}{b^m} \\ &= \frac{-c^{1-\lambda}}{b} \sum_{n=0}^{\infty} \sum_{q=0}^{n-\lambda} \frac{(\alpha)_{n-q-\lambda} x^{n-q-\lambda} a^q f_q(b, \beta)}{[n-q-\lambda] [q]} \end{aligned}$$

$$\text{where } f_q(b, \beta) = \sum_{m=0}^{\infty} \frac{(\beta m)_q}{b^m} \quad (2.3)$$

Hence, we obtain the series form as

$$Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda) = \frac{[-n] c^{1-\lambda}}{t} \sum_{q=0}^{n-\lambda} \frac{(\alpha)_{n-q-\lambda} x^{n-q-\lambda}}{[n-q-\lambda]} \times \frac{a^q f_q(b, \beta)}{[q]} \quad (2.4)$$

where $f_q(b, \beta)$ is defined in (2.3). The relation (2.2) may be written in the form

$$Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda) = \sum_{m=0}^{\alpha} \sum_{p=0}^{n-\lambda} \frac{(-1)(\alpha)_{n-\lambda} [n] c^{1-\lambda} x^{n-\lambda}}{[n-k] b^{m+1}} \times \frac{(-n+\lambda)_p (\beta m)_p a^p}{(1-\alpha-n+\lambda)_p [p] x^p} \quad (2.5)$$

Remarks: The series on the R.H.S. of (2.2) is convergent. If $b > 1$ when $1 = \lim_{m \rightarrow \alpha} \frac{f(x, a, m+1)}{f(x, a, m)}$ and

$$f(x, a, m) = {}_2F_1 \left[\begin{matrix} -n + \lambda & \beta m; & \frac{a}{x} \\ 1 - \alpha - n + \lambda \end{matrix} \right]$$

Note: For simplicity, $Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda)$ will be written as $Q_n(x; a, b)$ and $Q_n(x; a, 0)$ stands for the particular case when $b = 0$.

3. Convergence of $f_q(b, \beta)$. By applying D' Alembert's Ratio test, we have

$$\lim_{n \rightarrow \alpha} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \alpha} b \frac{(\beta_n)_q}{(\beta_n + \beta)_q} = b$$

Hence $f_q(b, \beta)$ is convergent. If $b > 1$ and divergent if $b < 1$. If $b = 1$ the limit of the n th terms is not zero. Hence the series is divergent.

4. Conditions for existence of $Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda)$

- (i) Either $b = 0$ or $b > 1$
- (ii) α, β are real numbers and n is a non-negative integer which are greater than λ , b be a non-negative integer.
- (iii) For the convergence of the polynomial se

$$Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda), \quad |t| < \min\{|x|^{-1}, |a|^{-1}\}$$

5. Nature of the polynomial $Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda)$

(i) Since $Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda) = (-1)^{n-\lambda} Q_n(-x; a, b, c, \lambda)$. This show that $Q_n(x; a, b)$ is an eve or odd Polynomial according as $n - \lambda$ is even or odd.

(ii) From the relation (2.2), it is clear that $Q_n(x; a, b) = Ax^{n-\lambda} + \pi_{n-\lambda-1}$

where $A = \frac{c^{1-\lambda}(\alpha)_{n-\lambda}}{(n-\lambda)(-b)}$ and $\pi_{n-\lambda-1}$ is a polynomial of

degree $(n - \lambda - 1)$. Thus $Q_n(x; a, b)$ is a polynomial of degree $n - \lambda$ in x .

(iii) $Q_\lambda(x; a, b) = \frac{c^{1-\lambda} \lambda}{1-b} = a$ constant. Thus, the polynomial will exist only for $n \geq \lambda$

6. Relationship between $Q_n(x, a, b)$ and other polynomials:

The set $\{Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda)\}$ defined by (2.1) is a generalization of the sequence of polynomials $\{D_n(x; a\lambda)\}_{m=0}^\infty$ gives by Karande and Thakare [3]. If we replace x by x/α , a by $1/\beta$, b by a and $c = 2$, then

$$\begin{aligned} \text{(i)} \quad \lim_{\min\{|\alpha|, |\beta|\} \rightarrow \infty} Q_n^{(\alpha, \beta)} \left(\frac{x}{\alpha}, \frac{1}{\beta}; a, 2, \lambda \right) \\ = \frac{2 \left(\frac{t}{2} \right)^\lambda e^{xt}}{e^t - a} = \sum_{r=0}^\alpha D_r(x; a, \lambda) \frac{t^r}{r} \end{aligned}$$

(ii) The polynomial $\{Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda)\}$ will be Bernoulli Polynomials given by Rai P.N. and Singh S.N. [4], $B_n(x)$ if we Replace x by x/α , a by $1/\beta$ and put $b = 1, \lambda = 1, c = 1$. So that

$$\begin{aligned} \lim_{\min\{|\alpha|, |\beta|\} \rightarrow \infty} Q_n^{(\alpha, \beta)} \left(\frac{x}{\alpha}, \frac{1}{\beta}, 1, 1, 1 \right) \\ = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^\infty B_n(x) \frac{t^n}{n} \end{aligned}$$

(iii) Similarly

$$\begin{aligned} \lim_{\min\{|\alpha|, |\beta|\} \rightarrow \infty} Q_n^{(\alpha, \beta)} \left(\frac{x}{\alpha}, \frac{1}{\beta}, -1, 2, 0 \right) \\ = \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^\infty E_n(x) \frac{t^n}{n} \end{aligned}$$

where $E_n(x)$ is the Euler polynomials, our polynomial becomes Eulerian if we replace x by $\frac{x}{\alpha}$, a by $\frac{1}{\beta}$ and β by $\frac{1}{\eta}$ and c by

$\left(1 + \frac{1}{\eta}\right)$; So that

$$\begin{aligned} \text{(iv)} \quad \lim_{\min\{|\alpha|, |\beta|\} \rightarrow \infty} Q_n^{(\alpha, \beta)} \left(\frac{x}{\alpha}, \frac{1}{\beta}, \frac{1}{\eta}, 1 + \frac{1}{\eta}, 0 \right) \\ = \frac{\left(1 + \frac{1}{\eta}\right) e^{xt}}{e^t + \frac{1}{\eta}} = \frac{(1-\eta)e^{xt}}{1-\eta e^t} = \sum_{n=0}^\infty \phi_n(x, \eta) \frac{t^n}{n} \end{aligned}$$

where $\phi_n(x, \eta)$ is Eulerian Polynomial defined by Carlitz [1]

(v) Similarly,

$$\lim_{\min\{|\alpha|, |\beta|\} \rightarrow \infty} Q_n^{(\alpha, \beta)} \left(\frac{x}{\alpha}, \frac{1}{\beta}, 1, 1, 1 \right) = \frac{te^{xt}}{e^t + 1} = \frac{1}{2} G_n(x)$$

where $G_n(x)$ is Genocchi Polynomial.

(vi) $\beta \rightarrow 0$ $Q_n^{(\alpha, \beta)} \left(0, \frac{1}{\beta}, 1, 1, 1 \right) = \beta_n$ (Bernoulli number)

(vii) $Q_n^{(\alpha, \beta)}_{\{|\alpha|, |\beta|\} \rightarrow \infty} \left(\frac{1}{\alpha}, \frac{1}{\beta}, -1, 2, 0 \right) = E_n$ (Euler's number)

(viii) $Q_n^{(\alpha, \beta)}_{\beta \rightarrow \infty} \left(0, \frac{1}{\beta}, -1, 1, 1 \right) = \frac{1}{2} G_n$ (Genocchi number)

7. Series Representation continued: The relation (2.1) may put in the form

$$\begin{aligned} \sum_{n=0}^\infty Q_n^{(\alpha, \beta)}(x; a, b, c, \lambda) \frac{t^n}{n} \\ = \frac{c \left(\frac{t}{c} \right)^\lambda}{\{(1-at)^{-\beta} - b\}} \sum_{n=0}^\infty \frac{(\alpha)_n x^n t^n}{n} \end{aligned}$$

$$= \sum_{r=0}^{\infty} Q_r^{(\alpha,\beta)}(o; a, b, c, \lambda) \frac{t^n}{\lfloor r} \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n t^n}{\lfloor n}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^n \frac{(\alpha)_{n-r} (x)^{n-r}}{\lfloor n-r} \frac{Q_r^{(\alpha,\beta)}(o, a, b, \lambda) t^n}{\lfloor r}$$

which is finally gives

$$Q_n^{(\alpha,\beta)}(x; a, b, c, \lambda) = \sum_{n=0}^{\infty} \binom{n}{r} Q_r^{(\alpha,\beta)}(o, a, b, c, \lambda) x^{n-r} (\alpha)_{n-r} \quad (7.1)$$

Alternatively, we may put (7.1) in the form

$$Q_n^{(\alpha,\beta)}(x; a, b) = \sum_{r=0}^n \binom{n}{r} x^r (\alpha)_r Q_{n-r}^{(\alpha,\beta)}(o; a, b, c, \lambda) \quad (7.2)$$

Remarks: We may define the Q_n numbers for $x = 0$ and $a = \frac{1}{\beta}$ by mean of generating relation

$$\sum_{n=0}^{\infty} Q_n \frac{t^n}{\lfloor n} = \frac{c \left(\frac{t}{c} \right)^\lambda}{\left(1 - \frac{t}{\beta} \right)^{-\beta} - b} \quad (7.3)$$

As such

$$\lim_{\beta \rightarrow \infty} \sum_{n=0}^{\infty} Q_n \frac{t^n}{\lfloor n} = \lim_{\beta \rightarrow \infty} \frac{c \left(\frac{t}{c} \right)^\lambda}{\left(1 - \frac{t}{\beta} \right)^{-\beta} - b} = \frac{c \left(\frac{t}{c} \right)^\lambda}{e^t - b} \quad (7.4)$$

Now, when $\lambda = b = 1$ and $c = 2$, Q_n becomes the Bernoulli number B_n [Rainville 1960 P – 200(4)], [5].

8. Recurrence Relations: Differential Recurrence relations from the definition.

$$\sum_{r=0}^{\infty} Q_n^{(\alpha,\beta)}(x; a, b, c, \lambda) \frac{t^n}{\lfloor n} = \frac{c \left(\frac{t}{c} \right)^\lambda (1 - xt)^{-\alpha}}{(1 - at)^{-\beta} - b}$$

Differentiating on both sides with respect to x .

$$\sum_{n=0}^{\infty} \frac{d}{dx} Q_n(x; a, b) \frac{t^n}{\lfloor n} = \frac{C \left(\frac{t}{c} \right)^\lambda (1 - xt)^{-\alpha}}{(1 - at)^{-\beta} - b} (\alpha t) (1 - xt)^{\lambda-1}$$

$$= \alpha \sum_{n=0}^{\infty} \sum_{r=0}^n \frac{Q_n(x; a, b)}{\lfloor n} x^r t^{n+r+1}$$

$$= \alpha \sum_{n=0}^{\infty} \sum_{r=0}^{n-1} \frac{Q_{n-r-1}(x; a, b) x^r t^n}{\lfloor n-r-1}$$

Hence,

$$\frac{d}{dx} Q_n(x; a, b) = \alpha \lfloor n \sum_{r=0}^{n-1} \frac{Q_{n-r-1}(x; a, b) - x^r}{\lfloor n-r-1} \quad (8.1)$$

Differentiating again with respect to x , we have

$$\sum_{n=0}^{\infty} \frac{d^2}{dx^2} Q_n(x; a, b) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1) Q_n(x; a, b)}{\lfloor n} t^{n+2} (1 - xt)^{-2}$$

$$= \sum_{n,r=0}^{\infty} \frac{\alpha(\alpha+1)}{\lfloor n} Q_n(x; a, b) \frac{(2)_r x^r t^{n+r+2}}{\lfloor r}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n-2} \frac{(\alpha)_2 (2)_r Q_{n-r-2}(x; a, b) x^r t^n}{\lfloor n-r-2} \frac{1}{\lfloor r}$$

Therefore

$$\frac{d^2}{dx^2} Q_n(x; a, b) = \sum_{r=0}^{n-2} \frac{(\alpha)_2 (2)_r Q_{n-r-2}(x; a, b) x^r \lfloor n}{\lfloor r \lfloor n-r-2} \quad (8.2)$$

Continuing the above process p times, we have

$$\frac{d^p}{dx^p} Q_n(x; a, b) = \sum_{r=0}^{n-p} \frac{(\alpha)_p (p)_r Q_{n-r-p}(x; a, b) \lfloor n x^r}{\lfloor n-r-p} \frac{1}{\lfloor r} \quad (8.3)$$

9. Mixed Recurrence Relations: Differentiating both sides of (2.1) with respect to x ,

$$\sum_{n=0}^{\infty} \frac{d}{dx} Q_n^{(\alpha,\beta)}(x; a, b) \frac{t^n}{\lfloor n} = \frac{c \left(\frac{t}{c} \right)^\lambda (1 - xt)^{-\alpha-1} \alpha t}{(1 - at)^{-\beta} - b}$$

$$= \alpha \sum_{n=0}^{\infty} \frac{Q_n^{(\alpha+1,\beta)}(x; a, b) t^{n+1}}{\lfloor n}$$

Equating coefficients of t^n on both sides,

$$\frac{d}{dx} Q_n^{(\alpha,\beta)}(x; a, b) = \alpha_n Q_{n-1}^{\alpha+1,\beta}(x; a, b) \quad (9.1)$$

Differentiating again w.r.t. x ,

$$\sum_{n=0}^{\infty} \frac{d^2}{dx^2} Q_n^{(\alpha,\beta)}(x; a, b) \frac{t^n}{\lfloor n}$$

$$= \frac{\alpha(\alpha+1) c \left(\frac{t}{c} \right)^\lambda (1 - xt)^{-\alpha-2} t^2}{(1 - at)^{-\beta} - b}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_2 Q_n^{(\alpha+2,\beta)}(x; a, b)}{\lfloor n} t^{n+2}$$

which on comparison of coefficients of t^n yields

$$\frac{d^2}{dx^2} Q_n^{(\alpha,\beta)}(x; a, b) = (\alpha)_2 \frac{\lfloor n}{\lfloor n-2} Q_{n-2}^{\alpha+2,\beta}(x; a, b) \quad (9.2)$$

Continuing on the same lines

$$\frac{d^r}{dx^r} Q_n^{(\alpha,\beta)}(x; a, b) = \frac{(\alpha)_r \lfloor n}{\lfloor n-r} Q_{n-r}^{\alpha+r,\beta}(x; a, b) \quad (9.3)$$

In particular, if $r = n - \lambda$, then

$$\frac{d^{n-\lambda}}{dx^{n-\lambda}} Q_n^{\alpha,\beta}(x; a, b) = \frac{(\alpha)_{n-\lambda} \lfloor n c^{1-\lambda}}{1-b} = \text{a constant} \quad (9.4)$$

This result shows that $Q_n^{(\alpha,\beta)}(x; a, b, c, \lambda)$ is polynomial of degree $n - \lambda$ in x ,

10. Mixed recurrence relations (continued):

(i) By definition

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n^{\alpha+1,\beta}(x; a, b) \frac{t^n}{[n]} &= \frac{c \left(\frac{t}{c}\right)^\lambda (1-xt)^{-\alpha-1}}{(1-at)^{-\beta} - b} \\ &= \frac{c \left(\frac{t}{c}\right)^\lambda (1-xt)^{-\alpha}}{(1-at)^{-\beta} - b} (1-xt)^{-1} \\ &= \sum_{n,m=0}^{\infty} \frac{Q_n(x; a, b)}{[n]} x^m t^{n+m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q_{n-m}(x; a, b) \cdot \frac{t^n}{[n-m]} x^m. \end{aligned}$$

Therefore

$$Q_n^{(\alpha+1,\beta)}(x; a, b) = \sum_{m=0}^n Q_{n-m}(x; a, b) x^m \frac{[n]}{[n-m]} \quad (10.1)$$

(ii) By definition

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n^{\alpha-1,\beta}(x; a, b) \frac{t^n}{[n]} &= \frac{c \left(\frac{t}{c}\right)^\lambda (1-xt)^{-\alpha+1}}{(1-at)^{-\beta} - b} \\ &= (1-xt) \sum_{n=0}^{\infty} Q_n(x; a, b) \frac{t^n}{[n]} \\ &= \sum_{n=0}^{\infty} Q_n(x; a, b) \frac{t^n}{[n]} - x \sum_{n=0}^{\infty} Q_n(x; a, b) \frac{t^{n+1}}{[n]} \end{aligned}$$

Thus, we get

$$Q_n^{(\alpha-1,\beta)}(x; a, b) = Q_n(x; a, b) - xnQ_{n-1}(x; a, b)$$

OR

$$Q_n(x; a, b) = Q_n^{(\alpha-1,\beta)}(x; a, b) + xnQ_{n-1}(x; a, b) \quad (10.2)$$

(iii) Again

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n^{(\alpha-2,\beta)}(x; a, b) \frac{t^n}{[n]} &= (1-xt)^2 \frac{c \left(\frac{t}{c}\right)^\lambda (1-xt)^{-\alpha}}{(1-at)^{-\beta} - b} \\ &= (1-2xt + x^2t^2) \sum_{n=0}^{\infty} Q_n(x; a, b) \frac{t^n}{[n]} \\ &= \sum_{n=0}^{\infty} Q_n(x; a, b) \frac{t^n}{[n]} - 2x \sum_{n=0}^{\infty} Q_n(x; a, b) \frac{t^{n+1}}{[n]} \\ &\quad + x^2 \sum_{n=0}^{\infty} Q_n(x; a, b) \frac{t^{n+2}}{[n]} \end{aligned}$$

Therefore

$$Q_n^{(\alpha-2,\beta)}(x; a, b) = Q_n(x; a, b) - 2xnQ_{n-1}(x; a, b) + n(n-1)x^2Q_{n-2}(x; a, b)$$

OR

$$Q_n(x; a, b) - 2xnQ_{n-1}(x; a, b) = Q_n^{(\alpha-2,\beta)}(x; a, b) - n(n-1)x^2Q_{n-2}(x; a, b) \quad (10.3)$$

$$\begin{aligned} \text{(iv)} \sum_{n=0}^{\infty} Q_n^{(\alpha,\beta)}(x; a, b, c, \lambda + 1) \frac{t^n}{[n]} &= \frac{c \left(\frac{t}{c}\right)^\lambda (1-xt)^{-\alpha}}{(1-at)^{-\beta} - b} (t/c) \\ &= \frac{1}{c} \sum_{n=0}^{\infty} Q_n(x; a, b) \frac{t^{n+1}}{[n]} \end{aligned}$$

Therefore,

$$cQ_n(x; a, b, c, \lambda + 1) = nQ_{n-1}(x; a, b) \quad (10.4)$$

In particular,

- (i) $2D_n(x; b, \lambda + 1) = nD_{n-1}(x, b, \lambda)$
- (ii) $2B_n(x) = nD_{n-1}(r, 1, 0)$
- (iii) $2E_n(x) = nD_{n-1}(x, -1, -1)$
- (iv) $4\eta\phi_n(x, \eta) = n(\eta - 1)D_{n-1}\left(x, \frac{1}{\eta}, -1\right)$

Similarly, we get

$$\text{(v)} c^2Q_n(x; a, b, c, \lambda + 2) = n(n-1)Q_{n-2}(x; a, b, c, \lambda) \quad (10.5)$$

$$\text{(vi)} c^rQ_n(x; a, b, c, \lambda + r) = \binom{r}{n}Q_{n-r}(x; a, b, c, \lambda)$$

where $\binom{r}{n} = n(n-1)\dots\dots(n-r+1)$

Further,

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n(x; a, b, c, \lambda - 1) \frac{t^n}{[n]} &= \frac{c \left(\frac{t}{c}\right)^\lambda (1-xt)^{-\alpha}}{(1-at)^{-\beta} - b} \left(\frac{t}{c}\right)^{-1} \\ &= c \sum_{n=0}^{\infty} Q_n(x; a, b) \frac{t^{n-1}}{[n]} \end{aligned}$$

Hence

$$nQ_{n-1}(x; a, b, c, \lambda - 1) = cQ_n(x; a, b) \quad (10.6)$$

Similarly,

$$\binom{2}{n}Q_{n-2}(x; a, b, c, \lambda - 2) = c^2Q_n(x; a, b) \quad (10.7)$$

$$\binom{r}{n}Q_{n-1}(x; a, b, c, \lambda - r) = c^rQ_n(x; a, b) \quad (10.8)$$

11. Contiguous Relations: The result (10.2) in the form of contiguous polynomials, Carlson B.C. [2], can be written as $Q_n^{\alpha,\beta}(x, a, b) = Q_n^{(\alpha-1,\beta)}(x; a, b) + xnQ_{n-1}^{\alpha,\beta}(x; a, b)$

Similarly, the results (10.4) can be written as

$$cQ_n^{\alpha,\beta}(x, a, b, c, \lambda) = nQ_n^{(\alpha,\beta)}(x; a, b)$$

eliminating the term $Q_{n-1}(x; a, b)$, we finally get

$$Q_n^{\alpha,\beta}(x; a, b, c, \lambda) = Q_n^{(\alpha-1,\beta)}(x; a, b, c, \lambda) + Q_n^{\alpha,\beta}(x; a, b, c, \lambda)cx \quad (11.1)$$

12. Some Interesting Results: From the definition

$$\sum_{n=0}^{\infty} Q_n^{(2\alpha,2\beta)}(x; a, b^2, c, 2\lambda) \frac{t^n}{[n]} \frac{c \left(\frac{t}{c}\right)^{2\lambda} (1-xt)^{-2\alpha}}{(1-at)^{-2\beta} - b^2}$$

$$\begin{aligned}
 &= \frac{1}{c} \cdot \frac{c \left(\frac{t}{c}\right)^\lambda (1-xt)^{-\alpha}}{(1-at)^{-\beta} - b} \times \frac{c \left(\frac{t}{c}\right)^\lambda (1-xt)^{-\alpha}}{(1-at)^{-\beta} + b} \\
 &= \frac{1}{c} \sum_{n,m=0}^{\infty} Q_n^{(\alpha,\beta)}(x; a, b, c, \lambda), Q_m^{(\alpha,\beta)}(x; a, -b, c, \lambda) \frac{t^{n+m}}{\underline{n} \underline{m}} \\
 &= \frac{1}{c} \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{Q_{n-m}^{(\alpha,\beta)}(x; a, b, c, \lambda) Q_m^{(\alpha,\beta)}(x; a, -b, c, \lambda) t^n}{\underline{n-m} \underline{m}} \\
 &\text{Equating coefficients of } t^n \text{ both sides, we have} \\
 &Q_n^{(2\alpha, 2\beta)}(x; a, b^2, c, 2\lambda) \\
 &= \frac{1}{c} \sum_{m=0}^n \binom{n}{m} Q_{n-m}^{(\alpha,\beta)}(x; a, b, c, \lambda) Q_m^{(\alpha,m)}(x; a, -b, c, \lambda) \\
 &= \frac{1}{c} \sum_{m=0}^n \binom{n}{m} Q_{n-m}(x; a, b) Q_m(x; a, -b)
 \end{aligned}$$

(12.1)

using the relation (12.1) the following condition obtained which are believed to be new on once.

REFERENCES

- [1] L. Carlitz, "Extended Bernoulli and Eulerian Numbers," *Duke Math. J.* Vol. 311, pp 667-689, 1964
- [2] B. C. Carlson, "Special functions of Applied mathematics," *Academic Press Newyork*, San Francisc London, 1977.
- [3] B. K Karande and N. K. Thakare, "On the unification of Bernoulli and Euler polynomials," *Ind. J. Pure appl. Math.* Vol. 6 No. I, pp 98-107, 1975.
- [4] P. N. Rai and S. N. Singh, "On a generalization of Euler Polynomials," *Bull. col. Math soc.* Vol. 72, pp. 229-232, 1980.
- [5] E. D. Rainville, "Special faneties Macmillan" Co. New York, 1960.
- [6] R. K. Singh, "Generalized special functions and Polynomials," *IJPEM*, ISSN 2348-3881, vol. 5 No. II, pp. 55-61, 2017.
- [7] R. K. Singh, "Generalized functions for the Jacobi polynomials and Laguerre polynomials" *IJMSEA*, ISSN 0973-9424, vol. 11 No. II, pp. 123-128, 2017.
- [8] R. K. Singh, "Expansion of generalized polynomial set $Q_m^k(x_m, y)$ in terms of Jacobi and Sister Celin's polynomials," *IJSER*, ISSN-2229-5518, Vol. 5, Issue 4, pp. 1408-1415, 2014.