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New Eighth Order Numerical Technique for Solving Nonlinear Algebraic Equations

Usamatu Usman[1*], Muhammad Mubarak Muhammad[2], Ibrahim Yusuf Inuwa[3], Aminu Haruna[4]

[1*, 2, 3, 4]Department of Mathematics and Statistics, Federal Polytechnic, Bauchi, Bauchi State, Nigeria

Abstract— *This paper presents new iterative method of order eight for solving nonlinear algebraic equations. The method was derived based on the Taylor's series expansion and Halley's method. The convergence analysis of the new method was discussed and it has eighth order of convergence. Several numerical examples were given and show that the new method is comparable with the well-known existing methods of the same order.*

Keywords— *Nonlinear equations, Taylor's Series, iterative method*

I. INTRODUCTION

The method of solving nonlinear equation $f(x) = 0$, have being discussed by large number of researchers. The most famous method discovered by Newton known as Newton's method can be derived from the first two terms of Taylor's series expansion of $f(x)$. That is

$$f(x) = f(x_0) + (x - x_0)f'(x_0) \tag{1}$$

Putting $f(x) = 0$ and solving for x in Eqn.(1) we have

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2}$$

Hence, we can write this in iterative form as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3}$$

The Newton's method has second order of convergence.

Many researchers try to improve Newton's method in order to get more accuracy and higher order convergence. For example, Weerakoon and Fernando (2000) modified the Newton's method using the trapezoidal rule to produce Newton trapezoidal method. Homeier (2005) modified Newton's method using the inverse function and also produces cubic convergence. Related researches have been conducted by Abbasbandy(2003), Frontini(2003), Ozban(2004),Chun(2006), He (2003) and Jayaraman(2016) and many more.

Halley (1964), extended the Newton's method to the third term of the Taylor's series expansion of $f(x)$. That is

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2}f''(x_0) \tag{4}$$

Putting $f(x) = 0$ and taking $(x - x_0)$ as common factor we get

$$0 = f(x_0) + (x - x_0)\left[f'(x_0) + \frac{(x-x_0)}{2}f''(x_0)\right] \tag{5}$$

Rearranging Eqn.(5) we have

$$x = x_0 - \frac{f(x_0)}{f'(x_0) + \frac{1}{2}(x-x_0)f''(x_0)} \tag{6}$$

from Eq.(2). we have $x = x_0 - \frac{f(x_0)}{f'(x_0)}$, upon substituting this in Eq.(6) we have

$$x = x_0 - \frac{2f(x_0)f'(x_0)}{2(f'(x_0))^2 - f(x_0)f''(x_0)} \tag{7}$$

Hence, we can write this in iterative form as

$$x = x_0 - \frac{2f(x_0)f'(x_0)}{2(f'(x_0))^2 - f(x_0)f''(x_0)} \tag{8}$$

The Halley's method has third order of convergence. Many researchers try to improve Halley's method either by increasing the order of convergence or by reducing the number of functions evaluation in iteration. For example, Noor (2007), Hafiz and Al-Goria (2012), Kumar et al. (2018) and many more. In this paper, we present new iterative method. We derived the method using the first three terms of Taylor's series expansion of $f(x)$ and use of the Halley's method within the expansion. The new introduced method has eighth order of convergence.

II. THE NEW METHOD

Consider the nonlinear equation of the type $f(x) = 0$ where $f(x)$ is a real function, sufficiently differentiable, defined on a real interval I.

For simplicity, assume that $\alpha \in I$ is a zero for $f(x)$, that is $f(\alpha) = 0$ and assume that x_0 is an initial guess sufficiently close to α. We obtain from Taylor's series expansion of the function $f(x)$ that

$$f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2}f''(x_0) + \frac{(x-x_0)^3}{6}f'''(x_0) = 0 \tag{9}$$

Reordering Eq.(9) gives

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{(x-x_0)^2}{2}\frac{f''(x_0)}{f'(x_0)} - \frac{(x-x_0)^3}{6}\frac{f'''(x_0)}{f'(x_0)} \tag{10}$$

Now, from according to Solaiman and Hashim (2018)

$$x - x_0 = -\frac{f(x_0)}{f'(x_0)} - \frac{2(f(x_0))^2 f'(x_0) f''(x_0)}{4(f'(x_0))^4 - 4f(x_0)\left(f'(x_0)\right)^2 f''(x_0) + (f(x_0))^2 (f''(x_0))^2} \tag{11}$$

Substituting Eq.(11) in Eq.[10] gives

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{(f(x_0))^2 f''(x_0)\left[4\left(f'(x_0)\right)^4 - 2f(x_0)\left(f'(x_0)\right)^2 f''(x_0) + (f(x_0))^2\left(f''(x_0)\right)^2\right]^2}{2\left(f'(x_0)\right)^3\left[-2\left(f'(x_0)\right)^2 + f(x_0)f''(x_0)\right]^4} + \frac{\left(f'(x_0)\right)^3\left[4\left(f'(x_0)\right)^4 - 2f(x_0)\left(f'(x_0)\right)^2 f''(x_0) + (f(x_0))^2\left(f''(x_0)\right)^2\right]^3 f'''(x_0)}{6\left(f'(x_0)\right)^4\left[-2\left(f'(x_0)\right)^2 + f(x_0)f''(x_0)\right]^6} \tag{12}$$

2.1 Algorithm: For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{13}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} -$$

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$$\frac{(f(y_n))^2 f''(y_n)\left[4(f'(y_n))^4-2f(y_n)(f'(y_n))^2 f''(y_n)+(f(y_n))^2(f''(y_n))^2\right]^2}{2(f'(y_n))^3\left[-2(f'(y_n))^2+f(y_n)f''(y_n)\right]^4}+$$
$$\frac{(f'(y_n))^3\left[4(f'(y_n))^4-2f(y_n)(f'(y_n))^2 f''(y_n)+(f(y_n))^2(f''(y_n))^2\right]^3 f'''(y_n)}{6(f'(y_n))^4\left[-2(f'(y_n))^2+f(y_n)f''(y_n)\right]^6} \quad (14)$$

III. CONVERGENCE ANALYSIS

Now, we shall discuss the convergence analysis of algorithm 2.1.

3.1 Theorem:

Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval I. If x_0 is sufficiently close to α, then the method defined by algorithm 2.1 is of eighth order of convergence.

3.1.1 Proof:

Consider α is a root of $f(x)$ and let $e_n = x_n - \alpha$ be the error at the nth iteration. By using Taylor's series about $x = \alpha$ we have:

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \cdots], \quad \text{Where } c_k = \frac{1}{k!}\frac{f^k(\alpha)}{f'(\alpha)}, \quad k = 2,3,4,\ldots \quad (15)$$

from Eqn.(15) we have

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \cdots] \quad (16)$$

Then from Eqns.(15) and (16) we have:

$$\frac{f(x_0)}{f'(x_0)} = e_n - c_2 e_n^2 - (2c_3 - 2c_2^2)e_n^3 - (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + \cdots \quad (17)$$

Using Eqn.(17) we can write y_n in algorithm 2.1 as

$$y_n = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + \cdots \quad (18)$$

Expanding $f(y_n)$, $f'(y_n)$, $f''(y_n)$ and $f'''(y_n)$ about α and using Eqn.(18) we get

$$f(y_n) = f(\alpha) + (y_n - \alpha)f'(\alpha) + \frac{(y_n-\alpha)^2}{2}f''(\alpha) + \frac{(y_n-\alpha)^3}{6}f'''(\alpha) + \cdots$$
$$= f'(\alpha)[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + \cdots] \quad (19)$$

$$f'(y_n) = f'(\alpha) + (y_n - \alpha)f''(\alpha) + \frac{(y_n-\alpha)^2}{2}f'''(\alpha) + \frac{(y_n-\alpha)^3}{6}f^{(4)}(\alpha) + \cdots$$
$$= f'(\alpha)[1 + 2c_2^2 e_n^2 + (4c_2c_3 - 4c_2^3)e_n^3 + (6c_2c_4 - 11c_2^2c_3 + 8c_2^4)e_n^4 + \cdots] \quad (20)$$

$$f''(y_n) = f''(\alpha) + (y_n - \alpha)f'''(\alpha) + \frac{(y_n-\alpha)^2}{2}f^{(4)}(\alpha) + \frac{(y_n-\alpha)^3}{6}f^{(5)}(\alpha) + \cdots$$
$$= f'(\alpha)\begin{bmatrix} 2c_2 + 6c_2c_3e_n^2 + (12c_3^2 - 12c_2^2c_3)e_n^3 \\ +(24c_2^2c_3 - 42c_2c_3^2 + 12c_2^2c_4 + 18c_2c_4)e_n^4 + \cdots \end{bmatrix} \quad (21)$$

$$f'''(y_n) = f'''(\alpha) + (y_n - \alpha)f^{(4)}(\alpha) + \frac{(y_n-\alpha)^2}{2}f^{(5)}(\alpha) + \frac{(y_n-\alpha)^3}{6}f^{(6)}(\alpha) + \cdots$$
$$= f'(\alpha)\begin{bmatrix} 6c_3 + 24c_2c_4e_n^2 + (48c_3c_4 - 48c_2c_4)e_n^3 \\ +(96c_2^3c_4 - 168c_2c_3c_4 + 72c_4^2 + 60c_2^2c_5)e_n^4 + \cdots \end{bmatrix} \quad (22)$$

Substituting Eqs.(19)-(22) into Eq.(14) in algorithm 2.1 we obtain

$$x_{n+1} = \alpha + (c_2^4c_4 - 2c_2^5c_3)e_n^8 + 0(e_n^9) \quad (23)$$

Implying that

$$e_{n+1} = (c_2^4c_4 - 2c_2^5c_3)e_n^8 + 0(e_n^9) \quad (24)$$

Hence algorithm 2.1 has at least eighth order of convergence

IV. NUMERICAL EXAMPLES

Consider the following test examples:

$f_1(x) = (x-1)^3 - 1$
$f_2(x) = x^3 - 10$
$f_3(x) = x^3 + 4x^2 - 10$
$f_4(x) = \cos x - x$
$f_5(x) = (\sin x)^2 - x^2 + 1$

We compare the new method with Newton's method, Halley's method, (NR) method proposed by Noor (2007) and modified Halley's method (MHM) proposed by Solaiman (2018).

Table 1 shows the number of iteration n such that the stopping criterion is satisfied, the approximate zero x_n and the computational order of convergence (COC).

TABLE 1.

Method	n	x_n	COC
	$f_1(x)$, $x_0 = 2.5$		
Newton	7	2	2
Halley	5	2	3
NR	3	2	6
MHM	3	2	6
NM	3	2	8
	$f_2(x)$, $x_0 = 2$		
Newton	5	2.154434690031884	2
Halley	4	2.154434690031884	3
NR	3	2.154434690031884	6
MHM	3	2.154434690031884	6
NM	2	2.154434690031884	8
	$f_3(x)$, $x_0 = 1$		
Newton	5	1.3652300134140969	2
Halley	4	1.3652300134140969	3
NR	4	1.3652300134140969	6
MHM	3	1.3652300134140969	6
NM	3	1.3652300134140969	8
	$f_4(x)$, $x_0 = 1.7$		
Newton	5	0.7390851332151607	2
Halley	5	0.7390851332151607	3
NR	3	0.7390851332151607	6
MHM	3	0.7390851332151607	6
NM	3	0.7390851332151607	8
	$f_5(x)$, $x_0 = 1$		
Newton	7	1.4044916482153413	2
Halley	5	1.4044916482153413	3
NR	4	1.4044916482153413	6
MHM	3	1.4044916482153413	6
NM	3	1.4044916482153413	8

V. CONCLUSION

In this paper we considered developing new iterative method of order eight for solving nonlinear equations. The method was derived using Taylor's series expansion and Halley's method. The order of convergence has been established and proved to be of the eighth order. Five examples were tested. It is observed that the method can be competitive to those well-known methods and also improve the existing methods.

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