International Journal of Multidisciplinary Research and Publications
ISSN (Online): 2581-6187

A New Approach to Linear Filtering and Prediction Based on High-Order Signal Models

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***Abstract*—*This research paper addresses the issue of filtering for signal models described by high-order vector difference equations (VDEs). The study is divided into two parts. The first part focuses on the filtering problem for a linear second-order VDE driven by white noise, while the second part extends these findings to high-order models of the same structure. The study develops a recursive equation for the filtered estimate based on the linear second-order model. The innovations approach is directly applied to the second-order model to derive a recursion for the filtered estimate. The resulting filter is defined as a second-order recursion that preserves the mathematical structure of the given model with innovations feedback loops. The study shows that the innovations satisfy a first-order recursion in terms of the filtered estimates and the measurements. The study formulates equations for the estimation of the filtered values and also determines the covariance matrices for the associated errors, based on the respective error values. This study presents the generalization of filtering results for high-order models of the same structure in the second part of the paper. The research considers a p^{th}-order vector difference equation (VDE) model with additive white noise and a linear combination of the signal process with additive white noise as the observation process. The study develops a one-stage prediction estimator for the p^{th}-order VDE signal model and presents the characterization of the innovations sequence in terms of the one-stage prediction estimates. The study also derives formulas for the estimator gains and demonstrates that the resulting estimator is a p^{th}-order system that preserves the form of the given model with innovations feedback loops. The study further shows that the well-known Kalman filter is a special case of these findings.*

***Keywords*—** *Filtering, prediction, Estimation, Signal Processing, Kalman Filter*

I. INTRODUCTION

This research paper presents a theory of recursive estimation for stochastic processes, represented as linear high-order vector difference equations (VDEs). This modeling approach has gained popularity in various fields, such as image processing and elastic and mechanical systems. The study develops a recursive equation for the one-stage prediction estimate based on a linear second-order VDE model. The study is structured into two primary parts. The first part focuses on deriving the filtered estimate based on the second-order VDE signal model, while the second part is dedicated to developing the one-stage prediction estimate based on a linear high-order model of the same structure. The innovations approach is applied directly to the assumed signal model in both cases. The study shows that the resulting estimator in both cases has the same mathematical structure as the given model with innovations feedback loops. Various methods of representing discrete-time stochastic processes, such as state-space realization, transfer functions, and vector difference equations, have been used in this research.

II. FILERING BASED ON SECOND-ORDER VDE MODEL

2.1. The conditional Mean

The signal model that was used by Iskanderani [11], with the same assumptions, is considered here. However, for sake of completeness, the basic assumptions concerning this model are found in the Appendix. The signal model is a linear second-order VDE given by:

$$x_{k+1} = A_k x_k + D_k x_{k-1} + \Gamma_k \omega_k \tag{1}$$

$$y_k = C_k x_k + E_k x_{k-1} + v_k \tag{2}$$

This research article focuses on the subsequent development of equations (1) and (2) and the assumptions mentioned in the Appendix. The study employs an innovations approach and the second-order vector difference equation (VDE) signal model to obtain a recursive equation for the filtered estimate, thereby developing a recursive equation for the conditional mean:

$$\hat{x}_{k+1|k+1} = E[x_{k+1}|Y_{k+1}] \tag{3}$$

where, $Y_{k+1} = \{y_1, y_2, \cdots, y_{k+1}\}$ The function $\hat{x}_{k|k}$ is referred to as the filtered estimate of x_k given Y_k. Define the set

$$\tilde{Y}_{k+1} \triangleq \{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{k+1}\} \tag{4}$$

where $\{\tilde{y}_{k+1}\}$ is called the innovations sequence of $\{y_k\}$ given by

$$\begin{aligned}\tilde{y}_{k+1} &= y_{k+1} - E[y_{k+1}|Y_k], \\ &= y_{k+1} - C_{k+1}\hat{x}_{k+1|k} - E_{k+1}\hat{x}_{k|k}\end{aligned} \tag{5}$$

with the initial condition

$$\tilde{y}_1 = y_1 - E[y_1] = y_1 - C_1\bar{x}_1 - E_1\bar{x}_0 \tag{6}$$

This research article explores the properties of the innovations sequence defined by equations (5) and (6) and their exploitation in subsequent analysis. The sequence has three distinct properties, namely a zero mean, independence of the set $\tilde{Y}_k$, and spanning of the same space by the sets Y_k and $\tilde{Y}_k$. The proof of these properties is discussed by Gevers [9]. In the subsequent development, the assumption is made that $\bar{x}_0 = \bar{x}_1 = 0$, and an approach is employed that utilizes the independence of the innovations for writing purposes as follows:

$$\hat{x}_{k+1|k+1} = E[x_{k+1}|Y_{k+1}] = E[x_{k+1}|Y_k] + E[x_{k+1}|\tilde{y}_{k+1}] = A_k\hat{x}_{k|k} + D_k E[x_{k-1}|Y_{k-1}] + D_k E[x_{k-1}|\tilde{y}_k] +$$

$$E[x_{k+1}|\tilde{y}_{k+1}] = A_k\hat{x}_{k|k} + D_k\hat{x}_{k-1|k-1} + D_kE[x_{k-1}|\tilde{y}_k] +$$
$$E[x_{k+1}|\tilde{y}_{k+1}] = A_k\hat{x}_{k|k} + D_k\hat{x}_{k-1|k-1} + D_kH_k^1\tilde{y}_k + H_{k+1}^2\tilde{y}_{k+1} \quad (7)$$

where the $n \times m$ gain matrices H_k^1 and H_k^2 are defined respectively by

$$H_k^1 \triangleq cov(x_{k-1}, \tilde{y}_k)[cov(\tilde{y}_k, \tilde{y}_k)]^{-1} \quad (8)$$
$$H_k^2 \triangleq cov(x_k, \tilde{y}_k)[cov(\tilde{y}_k, \tilde{y}_k)]^{-1} \quad (9)$$

It is interesting to recognize that equation (1.7) is a second-order recursion in terms of the filtered estimates which keeps the form of the given model. This simple observation has not previously been observed for higher-order models of the form considered in this paper. Given the filtered estimate $\hat{x}_{k|k}$, the one-stage prediction estimate can be written as:

$$\hat{x}_{k+1|k} = E[x_{k+1}|Y_k] = E[x_{k+1}|Y_k] + E[x_{k+1}|\tilde{y}_{k+1}] - E[x_{k+1}|\tilde{y}_{k+1}] = E[x_{k+1}|Y_{k+1}] - E[x_{k+1}|\tilde{y}_{k+1}] = \hat{x}_{k+1|k+1} - H_{k+1}^2\tilde{y}_{k+1} \quad (10)$$

2.2. *Gains Derivation*

Define the prediction and filtered error vectors respectively by

$$\tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1} \quad (11)$$
$$\tilde{x}_{k|k} = x_k - \hat{x}_{k|k} \quad .(1.12) \quad (12)$$

The innovations sequence $\tilde{y}_{k+1}$ can be written as

$$\tilde{y}_{k+1} = C_k\tilde{x}_{k+1|k} + E_k\tilde{x}_{k|k} + v_k \quad (13)$$

with Eq. (6) as its initial condition. Define also the covariance matrices by the relations

$$\Sigma_{k|k-1} \triangleq E[\tilde{x}_{k|k-1}\tilde{x}_{k|k-1}^T] \quad (14)$$
$$\Sigma_{k|k} \triangleq E[\tilde{x}_{k|k}\tilde{x}_{k|k}^T] \quad (15)$$
$$\Pi_{k|k-1} \triangleq E[\tilde{x}_{k-1|k-1}\tilde{x}_{k|k-1}^T] \quad (16)$$

It is easy to see that:

$$cov(\tilde{y}_k, \tilde{y}_k) \triangleq K_k = C_k\Sigma_{k|k-1}C_k^T + C_k\Pi_{k|k-1}^TE_k^T + E_k\Pi_{k|k-1}C_k^T + E_k\Sigma_{k-1|k-1}E_k^T + R_k \quad (17)$$

Here, K_k is an $m \times m$ positive- definite matrix, since R_k was assumed to be positive-definite matrix See Appendix for overall assumptions involved in this section of the paper. Next the gain matrix H_k^1 is evaluated as follows:

$$H_k^1 = cov(x_{k-1}, \tilde{y}_k)[cov(\tilde{y}_k, \tilde{y}_k)]^{-1} = [\Pi_{k|k-1}C_k^T + \Sigma_{k-1|k-1}E_k^T]K_k^{-1} \quad (18)$$

The gain matrix H_k^2 is computed in a similar way as

$$H_k^2 = cov(x_k, \tilde{y}_k)[cov(\tilde{y}_k, \tilde{y}_k)]^{-1} = [\Sigma_{k|k-1}C_k^T + \Pi_{k|k-1}^TE_k^T]K_k^{-1} \quad (19)$$

The gain matrices H_k^1 and H_k^2 therefore depend on the covariance matrices $\Sigma_{k|k-1}$, $\Pi_{k|k-1}$, and $\Sigma_{k|k}$. In the following section, recursive formulas for the covariance matrices are derived.

2.3. *Covariance Matrices*

In this section, the covariance matrices are derived. Subtracting Eq. (7) from Eq. (1) gives

$$\tilde{x}_{k+1|k+1} = A_k\tilde{x}_{k|k} + D_k\tilde{x}_{k-1|k-1} + \Gamma_k\omega_k - H_{k+1}^2\tilde{y}_{k+1} - D_kH_k^1\tilde{y}_k \quad (20)$$

And from Eq. (10) and Eq. (11)

$$\tilde{x}_{k+1|k+1} = \tilde{x}_{k+1|k} - H_{k+1}^2\tilde{y}_{k+1} \quad (21)$$

And from Eq. (20) and Eq. (21)

$$\tilde{x}_{k+1|k} = A_k\tilde{x}_{k|k-1} + D_k\tilde{x}_{k-1|k-1} + \Gamma_k\omega_k - (A_kH_k^2 + D_kH_k^1)\tilde{y}_k \quad (22)$$

Then from Eq. (15) ad Eq. (21)

$$\Sigma_{k|k} = E[(\tilde{x}_{k|k-1} - H_k^2\tilde{y}_k)(\tilde{x}_{k|k-1} - H_k^2\tilde{y}_k)^T] = \Sigma_{k|k-1} - H_k^2C_k\Sigma_{k|k-1} - H_k^2E_k\Pi_{k|k-1} \quad (23)$$

Next from Eq. (14) and Eq. (22) and some mathematical manipulations

$$\Sigma_{k+1|k} = E\left[\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T\right] = (D_k - L_kE_k)\left(\Sigma_{k-1|k-1}D_k^T + \Pi_{k|k-1}A_k^T\right) + (A_k - L_kC_k)\left(\Pi_{k|k-1}^TD_k^T + \Sigma_{k|k-1}A_k^T\right) + \Gamma_kQ_k\Gamma_k^T \quad (24)$$

where L_k is defined by the matrix relation

$$L_k \triangleq A_kH_k^2 + D_kH_k^1 \quad (25)$$

Finally from Eq. (16), Eq. (21) and Eq. (22)

$$\Pi_{k+1|k} = E[\tilde{x}_{k|k}\tilde{x}_{k+1|k}^T] = \Sigma_{k|k}A_k^T + [\Pi_{k|k-1}^T - H_k^2C_k\Pi_{k|k-1}^T - H_k^2E_k\Sigma_{k-1|k-1}]D_k^T \quad (26)$$

The recursive equations are initialized by (A.5) - (A.7) of the Appendix.

2.4. *Covariance Matrices*

The innovations can be represented in terms of the filtered estimates as follows

$$\tilde{y}_{k+1} + C_{k+1}D_kH_k^1\tilde{y}_k = -(C_{k+1}A_k + E_{k+1})\hat{x}_{k|k} - C_{k+1}D_k\hat{x}_{k-1|k-1} + y_{k+1} \quad (27)$$

with the initial condition (1.6). Hence, the innovations assure a first-order recursion forced by the filtered estimates.

2.5. Main Results

These results are summarized in the following theorem.
Theorem 1

The filter for (1.1) and (1.2), with the stated assumptions mentioned in the appendix, is given by

$$\hat{x}_{k+1|k+1} = A_k\hat{x}_{k|k} + D_k\hat{x}_{k-1|k-1} + D_kH_k^1\tilde{y}_k + H_{k+1}^2\tilde{y}_{k+1} \quad (28)$$

where $k = 1,2, \ldots$, and initialized by

$$\hat{x}_{0|0} = \bar{x}_0 \quad , \quad \hat{x}_{1|1} = \bar{x}_1 \quad (29)$$

The innovations satisfy

$$\tilde{y}_{k+1} + C_{k+1}D_kH_k^1\tilde{y}_k = y_{k+1} - (C_{k+1}A_k + E_{k+1})\hat{x}_{k|k} - C_{k+1}D_k\hat{x}_{k-1|k-1} \quad (30)$$

where $k = 2,3, \ldots$, and inintialized by the initial condition

$$\tilde{y}_1 = y_1 - C_1\bar{x}_1E_1\bar{x}_0 \quad (31)$$

The gain matrices H_k^1 and H_k^2 are given by Eq. (18) and Eq. (19). The associated covariances are given by Eq. (23) – Eq. (26).
In addition, the one-stage prediction estimate is given by

$$\hat{x}_{k+1|k} = \hat{x}_{k+1|k+1} - H_k^2\tilde{y}_{k+1} \quad (32)$$

III. Filering Based on High-Order Vde Model

3.1. Signal Model and Basic Assumptions

Here, the model for the signal being considered is expressed as a p'th-order linear VDE in the form of:

$$x_{k+1} = \sum_{j=1}^{p} A_k^jx_{k-j+1} + \Gamma_k\omega_k \quad (33)$$
$$y_k = \sum_{j=1}^{p} C_k^jx_{k-j+1} + v_k \quad (34)$$

Ahmed I. Iskanderani, "A New Approach to Linear Filtering and Prediction Based on High-Order Signal Models," *International Journal of Multidisciplinary Research and Publications (IJMRAP)*, Volume 5, Issue 11, pp. 151-155, 2023.

where $k = 1,2, \ldots$, and $x_1, x_0, x_{-1}, \ldots, x_{-p+2}$ are initial vectors, $\{x_k\}$ is an n-vector stochastic process. A_k^j , $j = 1, \ldots, p$ are real $n \times n$ matrices, Γ_k is $n \times r$ real matrix, $\{\omega_k\}$ is an r-vector zero mean gaussian white-noise process with covariance

$$E[\omega_k \omega_l^T] = Q_k \delta_{kl} \tag{35}$$

where Q_k is $r \times r$ matrix. $\{y_k\}$ is an m-vector output measurement process, C_k^j , $j = 1,2, \ldots, p$ are real $m \times n$ matrices, $\{v_k\}$ is an m-vector zero mean gaussian white-noise process with covariance

$$E[v_k v_l^T] = R_k \delta_{kl} \tag{36}$$

Assume the following:

(1) The input noise $\{w_k\}$ and the output noise $\{v_k\}$ are independent.

(2) The initial vectors $x_0, x_1, x_{-1}, \ldots$, are jointly gaussian random vectors with means

$$E[x_j] = \bar{x}_j \quad , j = -p + 2, -p + 3, \ldots, 0,1 \tag{37}$$

(3) The initial vectors are independent of $\{w_k\}$ and $\{v_k\}$.

(4) R_k is an $m \times m$ positive-definite matrix for each k.

3.2. The One-stage Prediction Estimator for the p'th-order System

It is desired to develop a recursive equation for the conditional mean

$$\hat{x}_{k+1|k} = E[x_{k+1}|Y_k] \tag{38}$$

where $Y_k = \{y_1, y_2, \ldots, y_k\}$ is the observation sequence. Define the set $\tilde{Y}_k = \{\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k\}$ where $\{\tilde{y}_k\}$ is the innovations sequence defined by

$$\tilde{y}_k = y_k - E[y_k|Y_{k-1}] \ , \ k = p, p + 1, \ldots, \tag{39}$$

with the initial conditions

$$\tilde{y}_m = y_m - \sum_{i=1}^{p} C_m^j \bar{x}_{m-j+1}^j , m = 1,2, \ldots, p - 1 \tag{40}$$

Without loss of generality, assume $\bar{x}_1 = \bar{x}_0 = \cdots = \bar{x}_{-p+2} = 0$ in the subsequent development.

Theorem 2

The one-stage prediction estimator for (2.1) and (2.2), with the stated assumptions, is given by

$$\hat{x}_{k+1|k} = \sum_{j=1}^{p} A_k^j \hat{x}_{k-j+1|k-j} + \sum_{i=1}^{p} G_k^i \tilde{y}_{k-i+1} \ , \ \ \mathrm{k} = 1,2, \ldots \tag{41}$$

with the initial conditions

$$\hat{x}_{j|j-1} = \bar{x}_j \ , \ j = -p + 2, -p + 3, \ldots, -1,0,1 \tag{42}$$

where the $n \times m$ gain matrices are given by

$$G_k^i \bar{!} \sum_{m=1}^{i} A_k^{p-m+1} E[x_{k-p+m} \tilde{y}_{k-p+i}^T] K_{k-p+i}^{-1} \tag{43}$$

$$K_j = E[\tilde{y}_j \tilde{y}_j^T] \tag{44}$$

where $i = 1,2, \ldots, p$ and $\tilde{y}_k$ is the innovations sequence, defined by Eq. (39), to be characterized in Theorem 3.

Proof:

$$\hat{x}_{k+1|k} = E[x_{k+1}|Y_k] = E[\sum_{j=1}^{p} A_k^j x_{k-j+1} + \Gamma_k \omega_k |Y_k] = A_k^1 E[x_k|\{\tilde{y}_k, Y_{k-1}\}] + A_k^2 E[x_{k-1}|\{\tilde{y}_k, \tilde{y}_{k-1}, Y_{k-2}\}] \ + \cdots + A_k^p E[x_{k-p+1}|\{\tilde{y}_k, \tilde{y}_{k-1}, \cdots, \tilde{y}_{k-p+1}, Y_{k-p}\}] = A_k^1 \hat{x}_{k|k-1} + A_k^1 E[x_k|\tilde{y}_k] \ + A_k^2 \hat{x}_{k-1|k-2} + A_k^2 E[x_{k-1}|\tilde{y}_k] + A_k^2 E[x_{k-1}|\tilde{y}_{k-1}] \ + \cdots + A_k^p \hat{x}_{k-p+1|k-p} + A_k^p E[x_{k-p+1}|\tilde{y}_k] \ + \cdots + A_k^p E[x_{k-p+1}|\tilde{y}_{k-p}] = \sum_{j=1}^{p} A_k^j \hat{x}_{k-j+1|k-j} + \{A_k^1 E[x_k|\tilde{y}_k] + A_k^2 E[x_{k-1}|\tilde{y}_k] \ + \cdots + A_k^p E[x_{k-p+1}|\tilde{y}_k]\} \ + \{A_k^2 E[x_{k-1}|\tilde{y}_{k-1}] + A_k^3 E[x_{k-2}|\tilde{y}_{k-1}] \ + \cdots + A_k^p E[x_{k-p+1}|\tilde{y}_{k-1}]\} + \cdots \ + A_k^p E[x_{k-p+1}|\tilde{y}_{k-p+1}] = \sum_{j=1}^{p} A_k^j \hat{x}_{k-j+1|k-j} + \{A_k^1 cov(x_k, \tilde{y}_k) + A_k^2 cov(x_{k-1}, \tilde{y}_k) \ + \cdots + A_k^p cov(x_{k-p+1}, \tilde{y}_k)\}\tilde{y}_k \ + \{A_k^2 cov(x_{k-1}, \tilde{y}_{k-1}) + A_k^3 cov(x_{k-2}, \tilde{y}_{k-1}) \ + \cdots + A_k^p cov(x_{k-p+1}, \tilde{y}_{k-1})\}\tilde{y}_{k-1} + \cdots \ + A_k^p cov(x_{k-p+1}, \tilde{y}_{k-p+1})\tilde{y}_{k-p+1} = \sum_{j=1}^{p} A_k^j \hat{x}_{k-j+1|k-j} + \sum_{i=1}^{p} G_k^i \tilde{y}_{k-i+1} \tag{45}$$

The gains G_k^i 's are defined by equations (43) and (44), where $i = 1,2, \ldots, p$. An interesting observation is that the estimator for one-stage prediction follows a p'th order recursion, maintaining the structure of the signal model, including feedback loops from innovations. The expressions for the innovations are given by equations (39) and (40). However, as the estimator is presented in terms of the one-stage prediction estimate, it would be beneficial to express the innovations using the same estimate. In the following section, Theorem 3 provides a derivation of this expression to enhance the estimator's utility.

3.3. Innovations

Theorem 3

The innovations sequence of the observation process Eq. (34) can be written as $p - 1$'st-order recurrence relation in terms of the one-stage prediction estimates as

$$\tilde{y}_k + \sum_{i=1}^{p-1} H_k^i \tilde{y}_{k-i} = y_k - \sum_{j=1}^{p} C_k^j \hat{x}_{k-j+1|k-j} , k = p, p + 1, \ldots \tag{46}$$

with the initial conditions

$$\tilde{y}_m = y_m - \sum_{j=1}^{p} C_m^j \bar{x}_{m-j+1} \ , m = 1,2, \ldots, p - 1 \tag{47}$$

where the gains H_k^i's are defined by

$$H_k^i \triangleq \sum_{m=i+1}^{p} C_k^m E[x_{k-m+1} \tilde{y}_{k-i}^T] E[\tilde{y}_{k-i} \tilde{y}_{k-i}^T] K_{k-i}^{-1} \tag{48}$$

Proof:

From the independence of the output noise and the observations sequence, the sequence $\tilde{y}_k$ can be written as

$$\tilde{y}_k = y_k - E[y_k|Y_{k-1}] = y_k - C_k^1 E[x_k|Y_{k-1}] - C_k^2 E[x_{k-1}|Y_{k-1}] \ - \cdots - C_k^p E[x_{k-p+1}|Y_{k-1}] = y_k - C_k^1 \hat{x}_{k|k-1} - C_k^2 E[x_{k-1}|\{\tilde{y}_{k-1}, Y_{k-2}\}] - \cdots \ - C_k^p E[x_{k-p+1}|\{\tilde{y}_{k-1}, \tilde{y}_{k-2}, \cdots, \tilde{y}_{k-p+1}, Y_{k-p}\}] = y_k - C_k^1 \hat{x}_{k|k-1} - C_k^2 \hat{x}_{k-1|k-2} - \cdots - C_k^p \hat{x}_{k-p+1|k-p} \ - C_k^2 E[x_{k-1}|\tilde{y}_{k-1}] - C_k^3 E[x_{k-2}|\tilde{y}_{k-1}] - \cdots - C_k^p E[x_{k-p+1}|\tilde{y}_{k-1}] \ - C_k^3 E[x_{k-2}|\tilde{y}_{k-2} - C_k^4 E[x_{k-3}|\tilde{y}_{k-2}] - \cdots - C_k^p E[x_{k-p+1}|\tilde{y}_{k-1}] \ - \cdots - C_k^p E[x_{k-p+1}|\tilde{y}_{k-p+1}] = y_k - \sum_{j=1}^{p} C_k^j \hat{x}_{k-j+1|k-j} - \sum_{i=1}^{p-1} H_k^i \tilde{y}_{k-i} \tag{49}$$

where

$$H_k^i \triangleq \sum_{m=i+1}^{p} C_k^m E[x_{k-m+1} \tilde{y}_{k-i}^T] E[\tilde{y}_{k-i} \tilde{y}_{k-i}^T]^{-1} \tag{50}$$

General recursive formulas for the associated covariance matrices will not be given here since the general formulas are extremely complicated. However, for small p the formulas can easily be found.

3.4. General Results

International Journal of Multidisciplinary Research and Publications
ISSN (Online): 2581-6187

The overall results are summarized in the following theorem.

Theorem 4

The one-stage prediction estimator for equations (33) and (34) is given by

$$\hat{x}_{k+1|k} = \sum_{j=1}^{p} A_k^j \hat{x}_{k-j+1|k-j} + \sum_{i=1}^{p} G_k^i \tilde{y}_{k-i+1} \ , k = 1,2,\ldots \quad (51)$$

with the initial condition

$$\hat{x}_{j|j-1} = \bar{x}_j \ , \ j = -p+2, -p+3, \ldots, -1,0,1 \quad (52)$$

The innovations are given by

$$\tilde{y}_k + \sum_{i=1}^{p-1} H_k^i \tilde{y}_{k-i} = y_k - \sum_{j=1}^{p} C_k^j \hat{x}_{k-j+1|k-j} \ , k = p, p+1, \ldots \quad (53)$$

with the initial conditions

$$\tilde{y}_m = y_m - \sum_{j=1}^{p} C_m^j \bar{x}_{m-j+1} \ , m = 1,2,\ldots,p-1 \quad (54)$$

The $n \times m$ gain matrices are given by

$$G_k^i = \sum_{m=1}^{i} A_k^{p-m+1} E[x_{k-p+m} \tilde{y}_{k-p+i}^T] K_{k-p+i}^{-1} \quad (55)$$

$$K_k = \sum_{j=1}^{p} \sum_{i=1}^{p} C_k^j E[\tilde{x}_{k-j+1|k-1} \tilde{x}_{k-i+1|k-1}] C_k^{iT} + R_k \quad (56)$$

where $i = 1,2,\ldots,p$, and

$$H_k^i = \sum_{m=i+1}^{p} C_k^m E[x_{k-m+1} \tilde{y}_{k-i}^T] E[\tilde{y}_{k-i} \tilde{y}_{k-i}^T] K_{k-i}^{-1} \quad (57)$$

3.5. Kalman Filter as A Special Case

If $p = 1$ the signal model (33) and (34) reduces to the conventional first- order state-variable model

$$x_{k+1} = A_k x_k + \Gamma_k \omega_k (2.26) y_k = C_k x_k + v_k \quad (58)$$

The one-stage prediction estimator (Kalman filter) for this model is immediately found by letting $p = 1$ in Theorem 4.

IV. Conclusion

In this theoretical paper, the filtering of stochastic processes through recursive methods is explored. Specifically, vector difference equations are utilized as signal models. The innovations method is directly applied to derive the filtered estimate for the second-order model and the one-stage prediction estimate for the general p'th-order signal model. For both these filtering scenarios, it is established that the filter dynamics can be described by a linear system that maintains the mathematical structure of the given model, including innovations feedback loops. While this property is well-known for the Kalman filter, it has not been recognized for higher-order models until now. Additionally, the innovations can be computed using a straightforward recurrence relation based on the filtered estimates. These filters possess a crucial recursive form, making them highly practical for processing measurements and obtaining digital computer-based estimates.

Appendix

Second-Order Signal Model and Basic Assumptions

The signal model is represented by a linear second-order VDE in the form of (1) and (2), where $k = 1,2,\ldots,$. The stochastic process $\{x_k\}$ is an n-vector, with x_0 and x_1 as initial vectors. Real n×n matrices A_k, and $D_k$are used, along with an r-vector zero-mean Gaussian white-noise process $\{\omega_k\}$ having a covariance of

$$E[\omega_k \omega_l^T] = Q_k \delta_{kl} \quad (A.1)$$

where, Q_k is an $r \times r$ matrix, and δ represents the Kronecker delta function. Additionally, an n×r matrix Γ_k is employed. The output measurement process $\{y_k\}$ is an m-vector, with real $m \times n$ measurement matrices C_k and E_k.

$\{v_l\}$ is an m-vector, zero mean gaussian white-noise process with covariance

$$E[v_k v_l^T] = R_k \delta_{kl} \quad (A.2)$$

Assume also the following:

(1) The input noise $\{\omega_k\}$ and the output noise $\{v_k\}$ are independent in the sense that

$$cov(\omega_k, v_l) = 0 \ for\ all\ \ k, l = 1,2,\ldots \quad (A.3)$$

(2) The initial vectors x_0 and x_1 are jointly gaussian random vectors with means

$$E[x_0] = \bar{x}_0 \quad E[x_1] = \bar{x}_1 \quad (A.4)$$

and covariances

$$E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T] = \Sigma_{0|0} \quad (A.5)$$

$$E[(x_1 - \bar{x}_1)(x_1 - \bar{x}_1)^T] = \Sigma_{1|0} \quad (A.6)$$

$$E[(x_0 - \bar{x}_0)(x_1 - \bar{x}_1)^T] = \Pi_{1|0} \quad (A.7)$$

(3) The initial vectors x_0 and x_1 are independent of $\{v_k\}$ and $\{\omega_k\}$ in the sense that

$$E[x_i \omega_k^T] = 0 \ , \quad for\ \ i = 0,1\ and\ for\ all\ k = 1,2,\ldots \quad (A.8)$$

$$E[x_i v_k^T] = 0 \ , \quad for\ \ i = 0,1\ and\ for\ all\ k = 1,2,\ldots \quad (A.9)$$

(4) R_k is an $m \times m$ positive-definite matrix for each k.

Conflict of Interest

The authors declare that they have no conflict of interest.

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