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A Survey of Definitions for Fractional Calculus

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Abstract*— In order to serve as a reference, this paper offers an overview of practical, well-known fractional calculus formulas.*

Keywords*— Fractional calculus, Fractional derivatives, Fractional operators.*

I. Introduction

In the mathematical discipline known as fractional calculus, arbitrary order integrals and derivatives are studied and used in various applications. Despite being a misnomer, the term "fractional" is still in usage today.

Calculating fractions may be thought of as both an old and new subject. It is an old issue because it has been developed up to the present day based on some theories by Leibniz (1695, 1697) and Euler (1730). However, since it has only been the subject of professional conferences and treatises since 1974, it might also be regarded as a novel issue.

Liouville, who conducted the initial thorough investigation of fractional calculus, was drawn to Abel's answer. Liouville was effective in applying his concepts to future theory issues in 1832. He discovered two formulations for the fractional derivative of the functions, but they were too constrained to hold up over time.

The first person to try to solve differential equations with fractional operators was Liouville. He defended the presence of a supplementary function in 1834. The complementary function was the subject of several papers written by Greatheed in 1839, and he was the first to draw attention to the complementary function's ambiguous nature. Riemann created his fractional integration theory in 1892. He derived from a generalisation of the Taylor series.

$$D^{-a}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau + \Psi(t) \quad (1)$$

He question of the existence of the complementary function Ψ(t) caused considerable confusion. Indeed, the present-day definition of fractional integration is "(1)," without a complementary function.

The study by Sonin seems to be the first effort that eventually led to what is now known as the Riemann-Liouville concept of fractional derivative. He began with the integral formula of Cauchy. From 1868 to 1872, Letnikov published four papers on this subject. Sonin's paper is expanded upon in his paper, which was published in 1872. The Cauchy formula's th derivative is given by

$$\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta. \quad (2)$$

Since there is no issue with generalisation to any value, $n! = \Gamma(n+1)$ the integrand in "(2)," no longer contains a pole but a branch point when is not an integer. Then, a branch cut would be necessary for a proper contour, which was mentioned in Sonin and Letnikov's work but was not incorporated.

In 1884, Laurent released his paper, which also included the Cauchy integral formula. In contrast to the closed circle of Sonin and Letnikov, his contour was an open circle on a Riemann surface. The definition was created using the contour integration technique.

$${}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau,$$
$$\operatorname{Re}\alpha > 0, \quad (3)$$

for integration to an arbitrary order. When $t > a$ in "(3)," we have Riemann definition "(1)," but without a complementary function. The most used version is setting $a = 0$

$${}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau,$$
$$\operatorname{Re}\alpha > 0. \quad (4)$$

This form of the fractional integral is often referred to as the Riemann-Liouville fractional integral. For the fractional integrals "(3)," and "(4)," we adopt the widely used notations ${}_aJ_t^\alpha$ and J_t^α respectively.

In 1892, Heaviside published a number of papers in which he showed how certain linear differential equations may be solved by the use of generalized operators. His methods, which have proved useful to engineers in the theory of transmission of electrical currents in cables, have been collected under the name Heaviside operational calculus.

Heaviside operational calculus is concerned with linear functional operators. He denoted the differentiation operator by the letter P and treated it as if it were a constant in the solution. For example, the heat equation in one dimension is

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}, \quad (5)$$

where a^2 is a constant and u is the temperature. If we let $\frac{\partial}{\partial t} = P$, then "(5)," becomes

$$D^2 u = a^2 Pu. \quad (6)$$

Gregory put the solution of "(5)," into symbolic operator form

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$$u(x,t) = Ae^{xaP^{1/2}} + Be^{-xaP^{1/2}}. \tag{7}$$

This is exactly what you would get if you solved "(6)," assuming P a constant.

However, Heaviside's clever applications were what really helped the theory of these generalised operators advance. In order to get the right outcomes, he increased the exponential in powers of $P^{1/2}$, where $P^{1/2} = d^{1/2}/dt^{1/2} = D^{1/2}$. In the theory of electrical circuits, Heaviside found frequent use for the operator $P^{1/2}$.

Although his results were accurate, he was unable to defend his methods, and it took a lot longer for Bromwich to do so in 1919.

There was only a little amount of published work on the fractional calculus between 1900 and 1970. Al-Bassam, Davis, Erdélyi, Hardy, Kober, Littlewood, Love, Osler, Riesz, Samko, Sneddon, Weyl, and Zygmund were some of the authors.

The National Science Foundation funded the first international conference on fractional calculus, which took place at the University of New Haven in Connecticut in 1974.

Theory were among the topics covered, all of which were highly enthralling. The first book by Oldham and Spanier addressing fractional calculus was released in 1974.

Springer-Verlag published the conference proceedings. Numerous eminent mathematicians were present. Among these giants were Askey, Mikolás, and a large number of the aforementioned mathematicians. Papers on the fractional calculus and extended functions, inequalities found by using the fractional calculus, and applications of the differential derivatives to probability.

In 1984, the University of Strathclyde in Glasgow, Scotland, held the second international conference on fractional calculus. The following people contributed to the proceedings: Heywood, Kalla, Lamb, Lowndes, Nishimoto, Rooney, and Srivastava.

At Nihon University in Tokyo, the third international conference took place in 1989. Al-Bassam, Bagley, Brychkov, Camoos, Gorenflo, Joshi, Kalla, Love, Mikolás, Nishimoto, Owa, Prudnikov, Ross, Samko, and Srivastava were a few of the numerous contributors.

Fractional derivatives provide an excellent tool for the description of memory [1], [2] and hereditary properties of various materials and processes. This is one of the advantages of fractional derivatives in comparison with classical integer-order models. Also, fractional calculus finds use in many other fields of science and engineering including control theory [3]-[5], fractals theory [6], optics [7], fluid flow [8], diffusion [9]-[10], electromagnetic theory [11]-[12], potential theory [20] and more. Interested readers in this topic can see also [8] and [21].

II. Fractional Derivatives and Integrals Other

The conventional fractional derivatives have many definitions. Fractional derivatives such as Grünwald-Letnikov, Riemann-Liouville, Caputo, Riesz, and Riesz-Feller are a few of them. Iterated derivatives with the same standard derivative are referred to as sequential derivatives.

A brief summary of the meanings and relationships of a few fractional derivatives is offered in the following subsections. for additional information and proofs of these relationships, see [8].

A. Riemann-Liouville Fractional Integral

The Riemann-Liouville fractional integral operator "(6),"of order $\alpha \geq 0$ of a function $f(t) \in L^1[a,b]$ and is defined as

$$\begin{cases} {}_aJ_t^\alpha f(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \alpha > 0, \\ {}_aJ_t^0 f(t) = f(t). \end{cases} \tag{8}$$

Some properties of ${}_aJ_t^\alpha f(t)$ are:
For $\alpha, \beta \geq 0$ and $\gamma > -1$

$${}_aJ_t^\alpha({}_aJ_t^\beta f(t)) = {}_aJ_t^{\alpha+\beta} f(t), \tag{9}$$

$${}_aJ_t^\alpha({}_aJ_t^\beta f(t)) = {}_aJ_t^\beta({}_aJ_t^\alpha f(t)), \tag{10}$$

$${}_aJ_t^\alpha (t-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\alpha+\gamma}. \tag{11}$$

Proofs of these properties and other relations involving ${}_aJ_t^\alpha f(t)$ are given in several text books (see for example [13] Ch.2 and [14] Ch.2).

B. *Grünwald-Letnikov Fractional Derivative*

Consider the Riemann-Liouville fractional integral "(6),". If the function $f(t)$ has $m+1$ continuous derivatives in the closed interval $[a,t]$, then for $m < \alpha < m+1$

$${}_aJ_t^\alpha f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{\alpha+k}}{\Gamma(\alpha+k+1)} + \frac{1}{\Gamma(\alpha+m+1)}\int_a^t (t-\tau)^{\alpha+m} f^{(m+1)}(\tau)d\tau, \tag{12}$$

by replacing each α by $(-\alpha)$ we get

$${}_aD_t^\alpha f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(-\alpha+m+1)}\int_a^t (t-\tau)^{-\alpha+m} f^{(m+1)}(\tau)d\tau \tag{13}$$

which is the fractional derivative of the Grünwald-Letnikov see [8] and [20].

The following relations are utilised to assess compositions of a derivative of integer-order with a derivative of arbitrary order.

$$\frac{d^n}{dt^n}({}_aD_t^\alpha f(t)) = {}_aD_t^{n+\alpha} f(t), \tag{14}$$

$${}_aD_t^\alpha\left(\frac{d^n}{dt^n} f(t)\right) = {}_aD_t^{n+\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-n}}{\Gamma(1+k-\alpha-n)}. \tag{15}$$

From equations "(14)," and "(15)," when $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$, it is found that

$$\frac{d^n}{dt^n}({}_aD_t^\alpha f(t)) = {}_aD_t^\alpha\left(\frac{d^n}{dt^n} f(t)\right) = {}_aD_t^{\alpha+n} f(t). \tag{16}$$

C. *Riemann-Liouville Fractional Derivative*

It should be noted that the Grünwald-Letnikov fractional derivative definition is based on the premise that the function must always be continuously differentiable. Consequently, the following definition for the Riemann-Liouville fractional derivative appeared.

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-\tau)^{-\alpha} f(\tau)d\tau, \tag{17}$$

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$0<\alpha<1, \quad t>a$.

or

$$ {}_aD_t^\alpha f(t)=\frac{d}{dt}({}_aJ_t^{1-\alpha} f(t)), \tag{18}$$

and for the general case $m-1<\alpha<m$

$$ {}_aD_t^\alpha f(t)=\frac{d^m}{dt^m}({}_aJ_t^{m-\alpha} f(t)), \quad m=1,2,\ldots \tag{19}$$

In particular, ([18] section 2.1) when $\alpha=0$

$$ {}_aD_t^0 f(t)=f(t), \tag{20}$$

and when $\alpha=n\in\mathbb{N}^+$

$$ {}_aD_t^n f(t)=\frac{d^n}{dt^n} f(t). \tag{21}$$

The following relations are utilised to evaluate compositions between a derivative of integer-order and a derivative of arbitrary order.

$$\frac{d^n}{dt^n}({}_aD_t^\alpha f(t))={}_aD_t^{n+\alpha} f(t), \tag{22}$$

$$ {}_aD_t^\alpha(\frac{d^n}{dt^n} f(t))={}_aD_t^{n+\alpha} f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)(t-a)^{k-a-n}}{\Gamma(1+k-\alpha-n)}. \tag{23}$$

which is the same relation as "(14)," and "(15),".

As before from equations "(22),"and "(23)," when $f^{(k)}(a)=0, \; k=0,1,\ldots,n-1,$ one obtains

$$\frac{d^n}{dt^n}({}_aD_t^\alpha f(t))={}_aD_t^\alpha(\frac{d^n}{dt^n} f(t))={}_aD_t^{\alpha+n} f(t). \tag{24}$$

Some properties of this definition are[4], [8]:

i. If p,q are two positive real numbers and $t>a$, then

$$ {}_aD_t^p({}_aJ_t^q f(t))={}_aD_t^{p-q} f(t). \tag{25}$$

ii. If $0\le k-1\le q<k,$ then

$$ {}_aJ_t^p({}_aD_t^q f(t))={}_aD_t^{q-p} f(t)-\sum_{j=1}^{k}\left[{}_aD_t^{q-j} f(t)\right]_{t=a}\frac{(t-a)^{p-j}}{\Gamma(1+p-j)}. \tag{26}$$

If the function $f(t)$ is $(n-1)$-times continuously differentiable and $f^{(n)}(t)$ is integrable in $[a,b]$, $t<b$ then for $0\le m-1\le\alpha<m\le n$

$$ {}_aD_t^\alpha f(t)=\sum_{j=0}^{m-1}\frac{f^{(j)}(a)(t-a)^{j-\alpha}}{\Gamma(1+j-\alpha)}+\int_a^t\frac{(t-\tau)^{-\alpha+m} f^{(m)}(\tau)}{\Gamma(m-\alpha)}d\tau. \tag{27}$$

D. Caputo Fractional Derivative

Unfortunately, the Riemann-Liouville method results in lower limit $t=a$ initial conditions with fractional derivatives, such as ${}_aD_t^{\alpha-1}f(a),\; {}_aD_t^{\alpha-2}f(a)$ etc.

Initial value problems with such initial conditions can be successfully solved mathematically, but there is no known physical interpretation for such types of initial conditions [8] Therefore, the definition of Caputo fractional derivative is used to overcome this problem. For an absolutely continuous function $f(t)$, Caputo fractional derivative is defined as

$$ {}_a^CD_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}\frac{d}{d\tau} f(\tau)d\tau, \tag{28}$$

Or

$$ {}_a^CD_t^\alpha f(t)={}_aJ_t^{1-\alpha}(\frac{d}{dt} f(t)), \tag{29}$$

and for the general case $m-1<\alpha<m$

$$ {}_a^CD_t^\alpha f(t)={}_aJ_t^{m-\alpha}(\frac{d^m}{dt^m} f(t)), \tag{30}$$

for absolutely continuous $\frac{d^m}{dt^m} f(t)$. In particular, ([13] section 2.4) when $\alpha=0$

$$ {}_a^CD_t^0 f(t)=f(t), \tag{31}$$

and when $\alpha=n\in\mathbb{N}^+$

$$ {}_a^CD_t^n f(t)=\frac{d^n}{dt^n} f(t). \tag{32}$$

Caputo definition also has the following advantages:

i. The initial conditions for FDEs and FPDEs with Caputo derivatives take on the same form as for integer-order differential equations, such as $f'(a), f''(a)$ etc., which have known physical interpretation.

ii. The Riemann-Liouville derivative of a constant does not equal zero, although the derivative calculated using the Caputo formulation does.

$$ {}_0D_t^\alpha c=\frac{ct^{-\alpha}}{\Gamma(1-\alpha)}, \qquad c \text{ is a constant} \tag{33}$$

Therefore, this definition was investigated by many authors (see for example [15] and [16]). It has the following properties:

i. If $m-1<\alpha\le m,$ $m\in\mathbb{N}$ and $f\in C_\mu^m, \;\; \mu\ge-1,$ then:

$$D_t^\alpha\left[J^\alpha f(t)\right]=f(t), \tag{34}$$

ii. $$J^\alpha\left[D_t^\alpha f(t)\right]=f(t)-\sum_{k=0}^{m-1} f^{(k)}(0)\frac{t^k}{k!}, \qquad t>0. \tag{35}$$

iii. $$D_t^\alpha t^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha}. \tag{36}$$

For more details on Caputo fractional derivative definition and its properties see [8, 17, 18].

E. Riesz Fractional Derivative

The Riesz fractional derivative R_x^α is defined as [22]

$$R_x^\alpha u(x)=-\frac{[D_+^\alpha u(x)+D_-^\alpha u(x)]}{2\text{Cos}(\alpha\pi/2)}, \quad 0<\alpha<2, \quad \alpha\ne1 \tag{37}$$

where $D_+^\alpha u(x)$ and $D_-^\alpha u(x)$ are the Weyl fractional derivatives

$$D_\pm^\alpha u(x)=\begin{cases}\pm\frac{d}{dx}I_\pm^{1-\alpha}u(x), & 0<\alpha<1\\ \frac{d^2}{dx^2}I_\pm^{2-\alpha}u(x), & 1<\alpha<2,\end{cases} \tag{38}$$

where $I_\pm^\beta$ denote the Weyl fractional integrals of order $\beta>0$, given by

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$$I_{+}^{\beta}u(x)=\frac{1}{\Gamma(\beta)}\int_{-\infty}^{x}(x-z)^{\beta-1}u(z)dz,$$

$$I_{-}^{\beta}u(x)=\frac{1}{\Gamma(\beta)}\int_{x}^{\infty}(z-x)^{\beta-1}u(z)dz. \tag{39}$$

When $\alpha=0$ the Weyl fractional derivative degenerates into the identity operator

$$D_{\pm}^{0}u(x)=Iu(x)=u(x). \tag{40}$$

For continuity we get

$$D_{\pm}^{1}u(x)=\pm\frac{d}{dx}u(x),\quad D_{\pm}^{2}u(x)=\frac{d^{2}}{dx^{2}}u(x). \tag{41}$$

Evidently, in the case $\alpha=2$ it takes the form of the second-derivative operator

$$R_{x}^{2}u(x)=\frac{d^{2}}{dx^{2}}u(x). \tag{42}$$

For the case $\alpha=1$ we have

$$R_{x}^{1}u(x)=\frac{d}{dx}Hu(x)=\frac{d}{dx}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(z)}{z-x}dz, \tag{44}$$

where H is the Hilbert transform and the integral is understood in the Cauchy principal value sense.

F. Riesz -Feller Fractional Derivative

The Riesz-Feller fractional operator for order for skewness , and for the one-variable function is defined as [19]

$${}_{x}D_{\theta}^{\alpha}u(x)=-[c_{L}(\alpha,\theta)\,{}_{-\infty}D_{x}^{\alpha}u(x)+c_{R}(\alpha,\theta)\,{}_{x}D_{+\infty}^{\alpha}u(x)],$$

Where

$${}_{-\infty}D_{x}^{\alpha}u(x)=\left(\frac{d}{dx}\right)^{m}\left[{}_{-\infty}I_{x}^{m-\alpha}u(x)\right], \tag{45}$$

$${}_{x}D_{+\infty}^{\alpha}u(x)=(-1)^{m}\left(\frac{d}{dx}\right)^{m}\left[{}_{x}I_{+\infty}^{m-\alpha}u(x)\right], \tag{46}$$

where $m\in\mathbb{N}, m-1<\alpha\le m$. In formula"(44)," coefficients $c_{L}(\alpha,\theta),\ c_{R}(\alpha,\theta)$,

(for $0<\alpha\le 2, \alpha\ne 1$, and $|\theta|\le min(\alpha,2-\alpha)$) have the following forms:

$$c_{L}(\alpha,\theta)=\frac{\sin\frac{(\alpha-\theta)\pi}{2}}{\sin(\alpha\pi)},\qquad c_{R}(\alpha,\theta)=\frac{\sin\frac{(\alpha+\theta)\pi}{2}}{\sin(\alpha\pi)}. \tag{47}$$

he left- and right-side of Weyl fractional integrals (Carpinteri and Mainardi, 1997; Podlubny, 1999; Samko., 1993) are what are meant by the fractional operators ${}_{-\infty}I_{x}^{\alpha}$ and ${}_{x}I_{+\infty}^{\alpha}$ and in expressions "(45),"and "(46),"

$${}_{-\infty}I_{x}^{\alpha}=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}\frac{u(z)}{(x-z)^{1-\alpha}}dz, \tag{48}$$

$${}_{x}I_{+\infty}^{\alpha}=\frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}\frac{u(z)}{(z-x)^{1-\alpha}}dz. \tag{49}$$

If $\alpha=0$, the Weyl fractional integrals are defined as the identity operator.

For $\alpha=1$, the representation (1.44) is not valid and has to be replaced by the formula

$${}_{x}D_{\theta}^{1}u(x)=[\cos(\theta\pi/2)D_{0}^{1}-\sin(\theta\pi/2)D]u(x), \tag{50}$$

where Feller initially noted in 1952 that the operator is related to the Hilbert transform:

$$D_{0}^{1}u(x)=\frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^{\infty}\frac{u(z)}{x-z}dz. \tag{51}$$

and D stays for the first derivative

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