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On Completely Monotonic Sequences

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Abstract**—***In this review article, we study recent developments on completely monotonic sequences. Necessary conditions, sufficient conditions, or necessary and sufficient conditions for a sequence to be a completely monotonic sequence or minimal completely monotonic sequence are introduced.*

Keywords**—***Completely monotonic sequence; Completely monotonic function; Minimal completely monotonic sequence.*

I. RECENT DEVELOPMENTS ON COMPLETELY MONOTONIC AND RELATED SEQUENCES

We first recall the notion of a completely monotonic sequence.

Definition 1 ([17]). *A sequence $\{\mu_n\}_{n=0}^{\infty}$ is called completely monotonic if*

$$(-1)^k\Delta^k\mu_n \geqq 0, \quad n,k\in\mathbb{N}_0,$$

where

$$\Delta^0\mu_n=\mu_n$$

and

$$\Delta^{k+1}\mu_n=\Delta^k\mu_{n+1}-\Delta^k\mu_n.$$

Here in Definition 1 and throughout the paper

$$\mathbb{N}_0:=\{0\}\cup\mathbb{N}$$

and $\mathbb{N}$ is the set of all positive integers.

For completely monotonic sequences, the following is the well-known Hausdorff's Theorem.

Theorem 2 ([17]). *A sequence $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic if and only if there exists a non‐decreasing and bounded function $\alpha(t)$ on $[0,1]$ such that*

$$\mu_n=\int_0^1 t^n d\alpha(t), \quad n\in\mathbb{N}_0.$$

The following property of completely monotonic sequences was established in [9].

Theorem 3 ([9]). *Suppose that the sequence $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic. Then, for $m,k\in\mathbb{N}_0$,*

$$\mu_m=(-1)^{k+1}\Delta^{k+1}\mu_m+\sum_{i=0}^{k}(-1)^i\Delta^i\mu_{m+1}.$$

Widder [23] defined a sub-class of the class of completely monotonic sequences as follows:

Definition 4 ([23]). *A sequence $\{\mu_n\}_{n=0}^{\infty}$ is called minimal completely monotonic if it is completely monotonic and if it will not be completely monotonic when μ_0 is replaced by a number less than μ_0.*

For the class of minimal completely monotonic sequences, Widder [23] proved the following result:

Theorem 5 ([23]). *A sequence $\{\mu_n\}_{n=0}^{\infty}$ is minimal completely monotonic if and only if there exists a non‐decreasing and bounded function $\alpha(t)$ on $[0,1]$ with*

$$\alpha(0)=\alpha(0+)$$

such that

$$\mu_n=\int_0^1 t^n d\alpha(t), \quad n\in\mathbb{N}_0.$$

We also recall the notion of a completely monotonic function, which is related to that of a completely monotonic sequence.

Definition 6 ([1]). *A function f is said to be completely monotonic on an interval I if f is continuous on I, has derivatives of all orders on I^o (the interior of I) and for all $n\in\mathbb{N}_0$*

$$(-1)^n f^{(n)}(x)\geqq 0, \quad x\in I^o.$$

The class of all completely monotonic functions on the interval I is denoted by $CM(I)$.

There is a rich literature on completely monotonic functions. For more recent work, see, for example, [2-16, 18-22].

There exists a close relationship between completely monotonic functions and completely monotonic sequences. For example, [23] showed the following

Theorem 7 ([23]). *Suppose that $f\in CM[a,\infty)$, then for any $\delta\geqq 0$, the sequence $\{f(a+n\delta)\}_{n=0}^{\infty}$ is completely monotonic.*

Suppose that $f\in CM[0,\infty)$. By Theorem 7, we know that the sequence $\{f(n)\}_{n=0}^{\infty}$ is completely monotonic.

On the other hand, for any given completely monotonic sequence $\{\mu_n\}_{n=0}^{\infty}$, we may ask whether there exists an interpolating function $f\in CM[0,\infty)$ such that

$$f(n)=\mu_n, \quad n\in\mathbb{N}_0.$$

For this interpolation question, Widder [23] established

Theorem 8 ([23]). *There exists a function $f\in CM[0,\infty)$ such that*

$$f(n)=\mu_n, \quad n\in\mathbb{N}_0$$

if and only if the sequence $\{\mu_n\}_{n=0}^{\infty}$ is minimal completely monotonic.

From Theorem 8, we see that the condition of minimal complete monotonicity is critical for a sequence $\{\mu_n\}_{n=0}^{\infty}$ to be interpolated by a completely monotonic function on the interval $[0,\infty)$.

The following result in [7] deals with a general interpolation question of completely monotonic sequences by completely monotonic functions.

Theorem 9 ([7]). *Suppose that the sequence $\{\mu_n\}_0^{\infty}$ is completely monotonic, then for any $\varepsilon\in(0,1)$, there exists a continuous interpolating function $f(x)$ on the interval $[0,\infty)$*

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such that $f|_{[0,\varepsilon]}$ *and* $f|_{[\varepsilon,\infty)}$ *are both completely monotonic and*

$$f(n) = \mu_n, \quad n \in \mathbb{N}_0.$$

Based on Theorem 9 the following result is also obtained in [7].

Theorem 10 ([7]). *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is completely monotonic. Then there exists a completely monotonic interpolating function* $g(x)$ *on the interval* $[1,\infty)$ *such that*

$$g(n) = \mu_n, \quad n \in \mathbb{N}.$$

It should be noted that under the condition of Theorem 10 we cannot guarantee that there exists a completely monotonic interpolating function $g(x)$ on the interval $[0,\infty)$ such that

$$g(n) = \mu_n, \quad n \in \mathbb{N}_0.$$

The following result [9] provides a necessary and sufficient condition for a sequence to be completely monotonic.

Theorem 11 ([9]). *A necessary and sufficient condition for the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *to be completely monotonic is that the sequence* $\{\mu_n\}_{n=1}^{\infty}$ *is completely monotonic, the series*

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1$$

converges and

$$\mu_0 \geqq \sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1.$$

In [22] the authors establish a sufficient condition, a necessary and sufficient condition for a sequence to be minimal completely monotonic as follows.

Theorem 12 ([22]). *Suppose that the sequence* $\{\mu_n\}_{n=1}^{\infty}$ *is completely monotonic and that the series*

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1$$

converges. Let

$$\mu_0^* := \sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1.$$

Then the sequence

$$\{\mu_0^*, \mu_1, \mu_2, \mu_3, \cdots\}$$

is minimal completely monotonic.

Theorem 13 ([22]). *A necessary and sufficient condition for the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *to be minimal completely monotonic is that the sequence* $\{\mu_n\}_{n=1}^{\infty}$ *is completely monotonic, the series*

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1$$

converges, and

$$\mu_0 = \sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1.$$

The following two results which give some necessary condition for a sequence to be completely monotonic or to be minimal completely monotonic are presented in [21].

Theorem 14 ([21]). *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is completely monotonic. Then, for any* $m \in \mathbb{N}$, *the series*

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_{m+1}$$

converges and

$$\mu_m = \sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_{m+1}.$$

Theorem 15 ([21]). *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is minimal completely monotonic. Then, for any* $m \in \mathbb{N}_0$, *the series*

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_{m+1}$$

converges and

$$\mu_m = \sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_{m+1}.$$

II. Conclusion

In this review article, we introduced recent developments on completely monotonic sequences. Necessary conditions, sufficient conditions, or necessary and sufficient conditions for a sequence to be a completely monotonic sequence, or minimal completely monotonic sequence are introduced.

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