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Review on Solving 2-D Poisson Problem by Finite and Compact Difference Methods

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Abstract— *Finite difference methods (FDM) are a popular class of numerical techniques for solving differential equations this is done by approximating derivatives with* finite differences*. Compact finite difference schemes enable us to produce more precise results with constrained grid sizes. The idea behind the derivation of the high-order compact scheme is to operate on the differential equations as an auxiliary relation to obtain finite difference approximations for high-order derivatives in the truncation error. In this paper, to generate approximate derivatives using finite differences, we shall discuss Taylor series expansions. For obtaining a more accurate numerical solution we will derive a compact* finite difference methods*. Finally, we will compare the two methods by solving two-dimensional Poisson equation in a rectangular domain.*

Keywords— *Compact finite difference schemes. Finite difference methods. Poisson equation. Taylor series.*

I. INTRODUCTION

One of the simplest and most established techniques for solving differential equations is the use of finite difference approximations for derivatives. Euler, L. (1768), previously knew it in one spatial dimension, and Runge, C. is likely the one who expanded it to dimension two (1908). The development of finite difference approaches in numerical applications was sparked by the appearance of computers in the early 1950s, which provided a practical framework for addressing challenging issues in science and industry [1]. Mickens, R. (2002) introduce non-traditional finite difference techniques that are helpful in the construction of differential equations. In his study, he discussed the precise finite difference scheme as well as the guidelines for creating nonstandard schemes and their use [2]. Sun, H. and Zhang, J. (2004) For the 2D convection diffusion problem, a sixth-order explicit finite difference discretization method based on the Richardson Extrapolation Technique and the Alternating Direction Implicit Method was developed [3]. Zhang, J., Geng, X., et.al. (2012) analyzed two approaches for enhancing the accuracy of the standard second order finite difference scheme in solving one dimensional elliptic partial differential equations. These two approaches are the fourth order compact difference scheme and Richardson extrapolation for the fourth order accuracy. They studied the truncation error of these two approaches. They provided both analytic and numerical evidence to clarify difference between two approaches [4]. Morsy, S.A. and Azab, M.S. (2012) in their research paper new finite difference method is introduced which is known as logarithmic finite difference method (log FDM). This method is improved for solving linear or nonlinear higher order partial differential equations [5]. Izadian, J.l., Ranjbar, N., et.al. (2013) Application of Generalized finite difference method for solving elliptic equation on irregular mesh are given. This method is used to 3-D Poisson's equation with Dirichlet boundary condition on irregular grids in a cuboid. By using Taylor series expansion and least squares, partial derivatives are approximated [6]. Zhai,S. et al. (2014) proposed sixth-order discretization method. To create difference schemes, they select a specific dual partition and use Lagrange interpolation and the Simpson integral formula. [7]. Zapata, M.U. and Balam,R.I., (2017) The two-dimensional Poisson equation is solved by implicit finite difference formulae in their research report using a novel family of high-order finite difference techniques. The Taylor series expansion and wave plane theory analysis are used to derive the implicit formulation, which is then built using a few tweaks to the conventional finite difference techniques. For the inner grid points, the approximations attain high order accuracy, and for the boundary grid points, up to ninth order accuracy [8]. Xia, H. and Gu, Y. (2021) made the first attempt to apply the generalized finite difference method (GFDM), a newly-developed meshless collocation method, for the numerical solutions of three-dimensional (3D) piezoelectric problems [9]. Jin, S. and Yue, Y. (2022) investigated time complexities of finite difference methods for solving the high-dimensional linear heat equation, the high-dimensional linear hyperbolic equation and the multiscale hyperbolic heat system with quantum algorithms [10].

For the conventional finite difference methods, a classical spatial discretization, such as the second-order central difference scheme, fails to approach the exact solution of most equations; in order to obtain a more accurate numerical solution, more nodes and smaller mesh sizes must be added, which would take up more storage space and processing time [11]. We must improve the numerical approximation's order of precision in order to obtain more accurate results for fixed mesh size, which necessitates expanding the grid's point stencil [12]. However, this leads to various issues, such as the challenging approaches to the boundary conditions, the approximation of the points next to the borders, and the expansion of the stiffness matrix's bandwidth. Higher-order numerical techniques should be used to solve numerous application issues accurately. For the

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aforementioned reasons, a compact finite difference technique is required to numerically solve several differential equations [13–14]. By creating high-order compact finite difference methods, one may calculate more accurate solutions with constrained grid sizes. Turkel, E. and Singer, I. (1998) have made significant contributions to this field [15]. In recent years, the high accuracy compact difference method has attracted more and more attention; [16–17]. Using a Taylor series expansion, Sari, M. et al (2010). developed a tenth-order finite difference scheme, proposed to solve one-dimensional advection–diffusion equation [18]. Gurarslan et al. (2014) The one-dimensional advection-diffusion equation was numerically solved using a sixth-order compact difference scheme in space and a fourth-order Runge-Kutta scheme in time. It has been shown to be incredibly precise in resolving the Pe 5 contamination transfer equation [19]. Cui, M.R. (2009) obtained a fully discrete implicit scheme based on the Grünwald-Letnikov discretization of the Riemann-Liouville derivative after approximating the second-order derivative with regard to space by the compact finite difference [20]. An effective and stable compact fourth-order technique for the phase field crystal equation was presented by Li, Y.B., and Kim in 2017 [21]. Li, L., Jiang, Z., and Yin, Z. (2018) developed an effective and practical compact finite difference approximation to a fourth-order method for resolving the linear one-dimensional convection-diffusion problem [17]. Overall, the creation and use of compact finite difference approaches for the numerical solution of the convection-diffusion equations has garnered considerable interest.

This paper is organized as follows, section 2 introduces forward, backward, and centered difference approximations of first and higher derivatives. Section 3 discusses compact finite difference for higher order derivatives. Section 4 contains numerical example to demonstrate high accuracy for compact finite schema. Concluding remarks are given in section 5.

II. Finite Difference Formulas

In this part, we shall introduce the idea of numerical differentiation. Remember that in order to obtain finite-divided-difference derivative approximations, we shall need Taylor series expansions. We shall provide first and higher derivative forward, backward, and center difference approximations. These estimations contained $o(h^2)$ errors. Their mistakes were inversely correlated with step size. This degree of precision is attributable to the amount of Taylor series terms that were kept in mind when these formulae were derived [1].

A. Forward Difference Approximation of the First Derivative

To determine a former value based on a current value, the Taylor series can be extended forward, as in

$$f\left(x_{i+1}\right)=f\left(x_i\right)+\frac{h}{1!}f^{(1)}(x_i)+\frac{h^2}{2!}f^{(2)}(x_i)+\ldots \qquad (1)$$

After taking the first derivative, truncating this equation and rearrange it produces.

$$f^{(1)}(x_i)=\frac{-f\left(x_{i+1}\right)+f\left(x_i\right)}{h}=\frac{\Delta f_i}{h} \qquad (2)$$

where h is referred to as the step size, which is the length of the interval that the approximation is made over, and Δf_i is referred to as the first forward difference. It uses data, hence the phrase "forward" difference at i and $i+1$ to estimate the derivative. where the error is $o(h)$.

There are several ways to construct the Taylor series from the forward divided difference to numerically approximate derivatives. For instance, the derivation of (2) may be used to build backward and centered difference approximations of the first derivative. Higher-order terms of the Taylor series can be used to create first derivative estimates that are more precise. Finally, second, third, and higher derivatives of all the aforementioned versions can be produced. Brief explanations of the methods used to develop some of these examples are provided below.

B. Backward Difference Approximation of the First Derivative

The Taylor series can be expanded backward to calculate a previous value on the basis of a present value, as in

$$f\left(x_{i-1}\right)=f\left(x_i\right)-\frac{h}{1!}f^{(1)}(x_i)+\frac{h^2}{2!}f^{(2)}(x_i)-\ldots \qquad (3)$$

Truncating this equation after the first derivative and rearranging yields

$$f^{(1)}(x_i)=\frac{f\left(x_i\right)-f\left(x_{i-1}\right)}{h}=\frac{\nabla f_i}{h} \qquad (4)$$

where ∇f_i is referred to as the first backward difference. It is termed a “backward” difference because it utilizes data at i and $i-1$ to estimate the derivative. where the error is $o(h)$.

C. Centered Difference Approximation of the First Derivative

A third way to approximate the first derivative is to subtract backward Taylor series from the forward Taylor series expansion to yield:

$$f\left(x_{i+1}\right)-f\left(x_{i-1}\right)=\frac{2h}{1!}f^{(1)}\left(x_i\right)+\frac{2h^3}{3!}f^{(3)}\left(x_i\right)+\ldots \qquad (5)$$

which can be solved for

$$f^{(1)}\left(x_i\right)=\frac{f\left(x_{i+1}\right)-f\left(x_{i-1}\right)}{2h}+\frac{2h^3}{3!}f^{(3)}\left(x_i\right)+\ldots \qquad (6)$$

Or

$$f^{(1)}(x_i)=\frac{f\left(x_{i+1}\right)-f\left(x_{i-1}\right)}{2h}+O\left(h^2\right) \qquad (7)$$

The first derivative is shown as a centered difference in the previous equation. In contrast to the forward and backward approximations, which were of the order of h, take note that the truncation error is of the order of h^2. It follows that the cantered difference is a more realistic depiction of the derivative, according to the Taylor series analysis. For instance, if we used a forward or backward difference to halve the step size, we would roughly halve the truncation error,

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however if we used a center difference, the error would be quartered.

D. Finite Difference Approximations of Higher Derivative

Besides first derivatives, the Taylor series expansion can be used to derive numerical estimates of higher derivatives. To do this, we write a forward Taylor series expansion for $f(x_{i+2})$ in terms of $f(x_i)$

$$f(x_{i+2}) = f(x_i) + \frac{2h}{1!} f^{(1)}(x_i) + \frac{(2h)^2}{2!} f^{(2)}(x_i) + \dots \quad (8)$$

Equation (1) can be multiplied by 2 and subtracted from (8) to give

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + h^2 f^{(2)}(x_i) + \dots \quad (9)$$

which can be solved for

$$f^{(2)}(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h) \quad (10)$$

This relationship is called the second forward finite divided difference. Similar manipulations can be employed to derive a backward version

$$f^{(2)}(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2} + O(h) \quad (11)$$

and a centered version

$$f^{(2)}(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2) \quad (12)$$

As was the case with the first-derivative approximations, the centered case is more accurate.

By using additional terms from the Taylor series expansion, the high-accuracy divided-difference formulas can be created. The forward Taylor series expansion, for instance, can be expressed as

$$f(x_{i+1}) = f(x_i) + \frac{h}{1!} f^{(1)}(x_i) + \frac{h^2}{2!} f^{(2)}(x_i) + \dots \quad (13)$$

which can be solved for

$$f^{(1)}(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{h}{2!} f^{(2)}(x_i) + O(h^2) \quad (14)$$

By eliminating the second- and higher-derivative terms from this finding, we were left with the following result:

$$f^{(1)}(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h) \quad (15)$$

In contrast to this approach, we now retain the second-derivative term by substituting the following approximation of the second derivative

$$f^{(2)}(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2} + O(h^2) \quad (16)$$

into (14) to yield

$$f^{(1)}(x_i) = \frac{f(x_{i+1}) + f(x_i)}{h} - \frac{h}{2!} \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2} + O(h^2) \quad (17)$$

or, by collecting terms

$$f^{(1)}(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} + O(h^2) \quad (18)$$

Note how adding the second derivative term increased precision to $o(h^2)$. The centering and backward formulae, as well as the approximations of the higher derivatives, may all be improved in a similar manner. The formulas are summarized in the below tables [22-23].

Table 1. Back Ward Divided Difference

First derivative	
$f^{(1)}(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$	$o(h)$
$f^{(1)}(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h}$	$o(h^2)$
$f^{(1)}(x_i) = \frac{11f(x_i) - 18f(x_{i-1}) + 9f(x_{i-2}) - 2f(x_{i-3})}{6h}$	$o(h^3)$
$f^{(1)}(x_i) = \frac{25f(x_i) - 48f(x_{i-1}) + 36f(x_{i-2}) - 16f(x_{i-3}) + 3f(x_{i-4})}{24h}$	$o(h^4)$
Second derivative	
$f^{(2)}(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}$	$o(h)$
$f^{(2)}(x_i) = \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2}$	$o(h^2)$
Third derivative	
$f^{(3)}(x_i) = \frac{f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})}{h^3}$	$o(h)$
$f^{(3)}(x_i) = \frac{5f(x_i) - 18f(x_{i-1}) + 24f(x_{i-2}) - 14f(x_{i-3}) + 3f(x_{i-4})}{2h^3}$	$o(h^2)$
Forth derivative	
$f^{(4)}(x_i) = \frac{f(x_i) - 4f(x_{i-1}) + 6f(x_{i-2}) - 4f(x_{i-3}) + f(x_{i-4})}{h^4}$	$o(h)$
$f^{(4)}(x_i) = \frac{3f(x_i) - f(x_{i-1}) + 26f(x_{i-2}) - 24f(x_{i-3}) + 11f(x_{i-4}) - 2f(x_{i-5})}{h^4}$	$o(h^2)$

Table 2. Forward Divided Difference

First derivative	
$f^{(1)}(x_i) = \frac{f(x_i) - f(x_{i+1})}{h}$	$o(h)$
$f^{(1)}(x_i) = \frac{3f(x_i) - 4f(x_{i+1}) + f(x_{i+2})}{2h}$	$o(h^2)$
$f^{(1)}(x_i) = \frac{11f(x_i) - 18f(x_{i+1}) + 9f(x_{i+2}) - 2f(x_{i+3})}{6h}$	$o(h^3)$
$f^{(1)}(x_i) = \frac{25f(x_i) - 48f(x_{i+1}) + 36f(x_{i+2}) - 16f(x_{i+3}) + 3f(x_{i+4})}{24h}$	$o(h^4)$

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Second derivative	
$f^{(2)}(x_i)=\dfrac{f(x_i)-2f(x_{i+1})+f(x_{i+2})}{h^2}$	$o(h)$
$f^{(2)}(x_i)=\dfrac{2f(x_i)-5f(x_{i+1})+4f(x_{i+2})-f(x_{i+3})}{h^2}$	$o(h^2)$
Third derivative	
$f^{(3)}(x_i)=\dfrac{f(x_i)-3f(x_{i+1})+3f(x_{i+2})-f(x_{i+3})}{h^3}$	$o(h)$
$f^{(3)}(x_i)=\dfrac{5f(x_i)-18f(x_{i+1})+24f(x_{i+2})-14f(x_{i+3})+3f(x_{i+4})}{2h^3}$	$o(h^2)$
Forth derivative	
$f^{(4)}(x_i)=\dfrac{f(x_i)-4f(x_{i+1})+6f(x_{i+2})-4f(x_{i+3})+f(x_{i+4})}{h^4}$	$o(h)$
$f^{(4)}(x_i)=\dfrac{3f(x_i)-14f(x_{i+1})+26f(x_{i+2})-24f(x_{i+3})+11f(x_{i+4})-2f(x_{i+5})}{h^4}$	$o(h^2)$

Table 3. Centered Divided Difference

First derivative	
$f^{(1)}(x_i)=\dfrac{f(x_{i+1})-f(x_{i-1})}{2h}$	$o(h^2)$
$f^{(1)}(x_i)=\dfrac{-f(x_{i+2})+8f(x_{i+1})-8f(x_{i-1})+f(x_{i-2})}{12h}$	$o(h^4)$
$f^{(1)}(x_i)=\dfrac{f(x_{i+3})-9f(x_{i+2})+45f(x_{i+1})-45f(x_{i-1})+9f(x_{i-2})-f(x_{i-3})}{60h}$	$o(h^6)$
Second derivative	
$f^{(2)}(x_i)=\dfrac{f(x_{i+1})-2f(x_i)+f(x_{i-1})}{h^2}$	$o(h^2)$
$f^{(2)}(x_i)=\dfrac{-f(x_{i+2})+16f(x_{i+1})-30f(x_i)+16f(x_{i-1})-f(x_{i-2})}{12h^2}$	$o(h^4)$
$f^{(2)}(x_i)=\dfrac{2f(x_{i+3})-27f(x_{i+2})+270f(x_{i+1})-490f(x_i)+270f(x_{i-1})-27f(x_{i-2})+2f(x_{i-3})}{60h^2}$	$o(h^6)$
Third derivative	
$f^{(3)}(x_i)=\dfrac{f(x_{i+2})-2f(x_{i+1})+2f(x_{i-1})-f(x_{i-2})}{2h^3}$	$o(h^2)$
$f^{(3)}(x_i)=\dfrac{-f(x_{i+3})+8f(x_{i+2})-13f(x_{i+1})+13f(x_{i-1})-8f(x_{i-2})+f(x_{i-3})}{8h^3}$	$o(h^4)$
Forth derivative	
$f^{(4)}(x_i)=\dfrac{f(x_{i+2})-4f(x_{i+1})+6f(x_i)-4f(x_{i-1})+f(x_{i-2})}{h^4}$	$o(h^2)$
$f^{(4)}(x_i)=\dfrac{-f(x_{i+3})+12f(x_{i+2})-39f(x_{i+1})+56f(x_i)-39f(x_{i-1})+12f(x_{i-2})-f(x_{i-3})}{6h^4}$	$o(h^4)$

III. Compact Finite Difference

The concept behind the derivation of the high-order compact scheme is to operate on the differential equations as an auxiliary relation to obtain finite difference approximations for high-order derivatives in the truncation error. This section will introduce a compact finite difference scheme (CFDS). De, A.k. and Eswaran, V. [24] defined compact schemes for simulating first order derivatives as

$$\sum_{k=-l}^{l}\beta_k\, f'_{i+k}=\frac{1}{h}\sum_{l=-m}^{m}a_l\, f_{i+l}, \tag{19}$$

With $\beta_0=1, \beta_k=\beta_{-k}$,where $l,m\in N$ By expanding the summations, the scheme given in the above equation can be expressed as

$$\beta_l\, f'_{i-l}+\cdots+\beta_1\, f'_{i-1}+f'_i+\beta_1\, f'_{i+1}+\cdots+\beta_l\, f'_{i+l}=\frac{1}{h}(a_{-m}f_{i-m}+\cdots+a_{-1}f_{i-1}+a_0 f_i+a_1 f_{i+1}+\cdots+a_m f_{i+m}) \tag{20}$$

The left-hand side (LHS) of (20) involves $2l+1$ derivative values while the right-hand side (RHS) has $2m+1$ node stencil. The scheme given in (20) is restricted to $l\le 2$ and $m\le 3$ because of the computational complexity in the use of implicit schemes. Any scheme can achieve the highest formal order of accuracy by increasing the value of l,m or both. By rewriting the RHS components of (20) into second order accurate centered finite differences, the centered compact schemes may be created. In particular (20) for $l=2$ and $m=3$ reduces to

$$\beta\, f'_{i-2}+\alpha\, f'_{i-1}+f'_i+\alpha\, f'_{i+1}+\beta\, f'_{i+2}=a\frac{f_{i+1}-f_{i-1}}{2h}+b\frac{f_{i+2}-f_{i-2}}{4h}+c\frac{f_{i+3}-f_{i-3}}{6h} \tag{21}$$

By substituting the first derivative in (21) with its corresponding higher-order derivative, similar centered compact schemes can be derived (21). The centered CFDS for $l=2$ and $m=3$ for an approximation of the second derivative is given by [25]

$$\beta f''_{i-2}+\alpha f''_{i-1}+f''_i+\alpha f''_{i+1}+\beta f''_{i+2}= a\frac{f_{i+1}-2f_i+f_{i-1}}{h^2}+b\frac{f_{i+2}-2f_i+f_{i-2}}{4h^2}+c\frac{f_{i+3}-2f_i+f_{i-3}}{9h^2} \quad (22)$$

where a,b,c,α and β are a few constant coefficients. These coefficients were calculated by the following Taylor series expansions of the terms in (22) are used

$$f_{i+1}=f_i+hf_i^{(1)}+\frac{h^2}{2}f_i^{(2)}+\frac{h^3}{3!}f_i^{(3)}+\frac{h^4}{4!}f_i^{(4)}+\frac{h^5}{5!}f_i^{(5)}+\frac{h^6}{6!}f_i^{(6)}+\frac{h^7}{7!}f_i^{(7)}+O\left(h^8\right),$$

$$f_{i-1}=f_i-hf_i^{(1)}+\frac{h^2}{2}f_i^{(2)}-\frac{h^3}{3!}f_i^{(3)}+\frac{h^4}{4!}f_i^{(4)}-\frac{h^5}{5!}f_i^{(5)}+\frac{h^6}{6!}f_i^{(6)}-\frac{h^7}{7!}f_i^{(7)}+O\left(h^8\right),$$

$$f_{i+2}=f_i+2hf_i^{(1)}+\frac{2^2h^2}{2}f_i^{(2)}+\frac{2^3h^3}{3!}f_i^{(3)}+\frac{2^4h^4}{4!}f_i^{(4)}+\frac{2^5h^5}{5!}f_i^{(5)}+\frac{2^6h^6}{6!}f_i^{(6)}+\frac{2^7h^7}{7!}f_i^{(7)}+O\left(h^8\right),$$

$$f_{i-2}=f_i-2hf_i^{(1)}+\frac{2^2h^2}{2}f_i^{(2)}-\frac{2^3h^3}{3!}f_i^{(3)}+\frac{2^4h^4}{4!}f_i^{(4)}-\frac{2^5h^5}{5!}f_i^{(5)}+\frac{2^6h^6}{6!}f_i^{(6)}-\frac{2^7h^7}{7!}f_i^{(7)}+O\left(h^8\right),$$

$$f_{i+3}=f_i+3hf_i^{(1)}+\frac{3^2h^2}{2}f_i^{(2)}+\frac{3^3h^3}{3!}f_i^{(3)}+\frac{3^4h^4}{4!}f_i^{(4)}+\frac{3^5h^5}{5!}f_i^{(5)}+\frac{3^6h^6}{6!}f_i^{(6)}+\frac{3^7h^7}{7!}f_i^{(7)}+O\left(h^8\right),$$

$$f_{i-3}=f_i-3hf_i^{(1)}+\frac{3^2h^2}{2}f_i^{(2)}-\frac{3^3h^3}{3!}f_i^{(3)}+\frac{3^4h^4}{4!}f_i^{(4)}-\frac{3^5h^5}{5!}f_i^{(5)}+\frac{3^6h^6}{6!}f_i^{(6)}-\frac{3^7h^7}{7!}f_i^{(7)}+O\left(h^8\right),$$

$$f_{i+1}^{(2)}=f_i^{(2)}+hf_i^{(3)}+\frac{h^2}{2}f_i^{(4)}+\frac{h^3}{3!}f_i^{(5)}+\frac{h^4}{4!}f_i^{(6)}+\frac{h^5}{5!}f_i^{(7)}+\frac{h^6}{6!}f_i^{(8)}+O\left(h^7\right),$$

$$f_{i-1}^{(2)}=f_i^{(2)}-hf_i^{(3)}+\frac{h^2}{2}f_i^{(4)}-\frac{h^3}{3!}f_i^{(5)}+\frac{h^4}{4!}f_i^{(6)}-\frac{h^5}{5!}f_i^{(7)}+\frac{h^6}{6!}f_i^{(8)}+O\left(h^7\right),$$

$$f_{i+2}^{(2)}=f_i^{(2)}+2hf_i^{(3)}+\frac{2^2h^2}{2}f_i^{(4)}+\frac{2^3h^3}{3!}f_i^{(5)}+\frac{2^4h^4}{4!}f_i^{(6)}+\frac{2^5h^5}{5!}f_i^{(7)}+\frac{2^6h^6}{6!}f_i^{(8)}+O\left(h^7\right),$$

$$f_{i-2}^{(2)}=f_i^{(2)}-2hf_i^{(3)}+\frac{2^2h^2}{2}f_i^{(4)}-\frac{2^3h^3}{3!}f_i^{(5)}+\frac{2^4h^4}{4!}f_i^{(6)}-\frac{2^5h^5}{5!}f_i^{(7)}+\frac{2^6h^6}{6!}f_i^{(8)}+O\left(h^7\right),$$

Substituting above equations into(22) and rearranging gives

$$\left(1+2\alpha+2\beta-a-b-c\right)f''_i+\left(2\alpha\frac{1}{2}+2\beta\frac{2^2}{2!}-\frac{2a}{4!}-\frac{2^4b}{2(4!)}-\frac{2\left(3^4\right)c}{9(4!)}\right)h^2f_i^{(4)}$$
$$+\left(2\alpha\frac{1}{4!}+2\beta\frac{2^4}{4!}-\frac{2a}{6!}-\frac{2^6b}{2(6!)}-\frac{2\left(3^6\right)c}{9(6!)}\right)h^4f_i^{(6)}+\left(2\alpha\frac{1}{6!}+2\beta\frac{2^6}{6!}-\frac{2a}{8!}-\frac{2^8b}{2(8!)}-\frac{2\left(3^8\right)c}{9(8!)}\right)h^6f_i^{(8)}+o\left(h^7\right)=0$$

The following system of equations is created by setting the coefficients of the previously mentioned equation to zero.

$$\begin{cases}(1+2\alpha+2\beta-a-b-c) &= 0,\\ 12(\alpha+2^2\beta)-a-2^2b-3^2c &= 0,\\ 30(\alpha+2^4\beta)-a-2^4b-3^4c &= 0,\\ 56(\alpha+2^6\beta)-a-2^6b-3^6c &= 0\\ 90(\alpha+2^8\beta)-a-2^8b-3^8c &= 0\end{cases} \quad (24)$$

Obviously, we can form different subsystems by extracting some or all of the equations from (24) which helps us to determine the values of a,b,c,α and β. For instance, a system the first two equations together have five unknowns, which results in a three-parameter family with an unlimited number of solutions. On the other hand, taking into account the first three equations result in an undecided system that can be resolved by resolving two of the five unknowns. The remaining equations can be used to create two further systems. Lele [26] claimed that by resolving systems by multiple CFDS of varying orders of accuracy may be created (24). In fact, we have the following:

1. If a,b,c,α and β satisfy the first equation of (24) only, then substituting these constants into formula (22) gives a compact scheme with second-order accuracy.
2. If the first two equations of (24) are satisfied by a,b,c,α and β, then a scheme with fourth-order accuracy is obtained.
3. If a,b,c,α and β satisfy the first three equations of (24), then the compact scheme obtained by replacing these constants into (22) is of order six.

If you keep doing this, you may get CFDSs with an accuracy of up to tenths of an order. It is encouraging to note that the accuracy of the resulting CFDS increases by two orders of magnitude every time a new equation is introduced to the preceding system [26].

IV. Numerical Experiments

We perform numerical tests and comparisons to evaluate the calculated accuracy attained by the compact and central schemes. Partial differential equations, such as the Poisson and Laplace equations, have numerous applications in a variety of disciplines, including computational fluid dynamics, structural mechanics, theoretical physics, etc.

$$\Delta u(x,y)=f(x,y), x\in\Omega \quad (25)$$

where Ω is a two-dimensional rectangular domain with Dirichlet boundary conditions defined as $u=\frac{\partial u}{\partial n}=0,(x,y)\in\Gamma$, and Δ is the Laplacian operator. When $f(x,y)=0$, (25) becomes the Laplace equation.

The Poisson equation is frequently used to describe equilibrium phenomena for a wide range of variables, including pressure, water surface elevation, temperature, and concentration [8]. The numerical solution of the Poisson equation is crucial for the computational simulation of the corresponding applied problems. In the last several decades, a lot of work has been put into creating numerical algorithms that solve the Poisson equation accurately.

Example 4.1: Defined Laplace equations by two diminutions with the Dirichlet boundary conditions as the following [27]

$$f_{xx}+f_{yy}=0,\ 0\le x\le 1,\quad 0\le y\le 1$$
$$f(x,0)=e^{-(2x)},\ f(x,1)=e^{-2x}cos(2),$$
$$f(0,y)=cos(2y),\ f(1,y)=e^{-2}cos(2y).$$

Now, we swear the region R for a finite number of rectangular elements. we choose of step lengths $h=0.25$ in x-axis and $k=0.25$ in y-axis. The exact solution of the above problem on the region is $f_{exact}(x,y)=e^{-2x}cos(2y)$.

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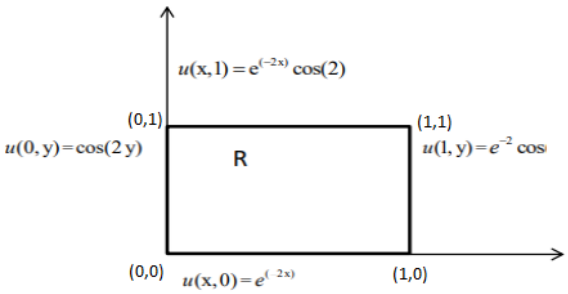

Fig.1. Boundary conditions in the region R

This Problem of Laplace Equation problem is solved by using the finite difference method by two different operators (five and nine) points and assuming step lengths $h = k = 0.25$ Then:

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

This is equation called Standard Five Points Equation (SFPE)

And

$$u_{i,j} = \frac{1}{20}\left[4\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right) + u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1}\right]$$

This is equation called Nine Points Equation (NPE)

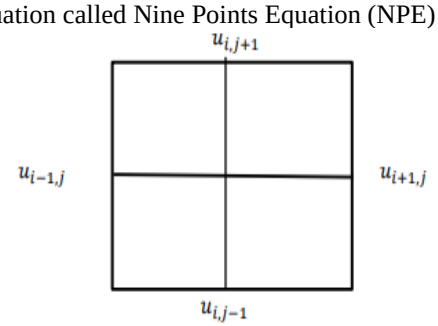

Fig. 2. Standard Five Points

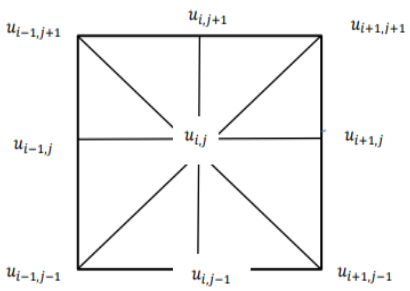

Fig.3. Nine Points

Table 4. Comparison of error between the (five and nine) points finite difference method on Example 4.1

Nodes Interior	Finite Difference Method		Exact Solution
	Five points	**Nine points**	
7	0.1972	0.1958	0.1958
8	0.3252	0.3228	0.3228
9	0.5348	0.5323	0.5322
12	0.3200	0.3277	0.3277
13	0.2010	0.1987	0.1987
14	0.1220	0.1206	0.1205
17	0.0164	0.0158	0.0157
18	0.0271	0.0260	0.0260
19	0.0438	0.0429	0.0429

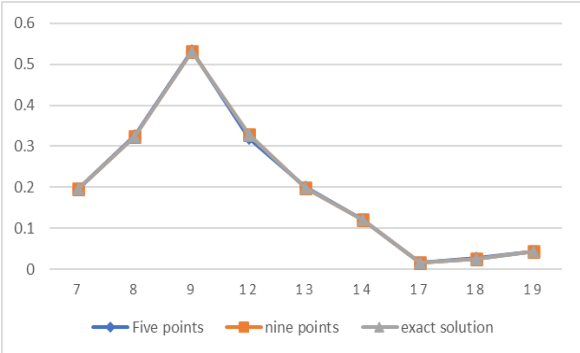

Fig.4. Comparison Between the results.

From the table 4, the Finite difference method by the operator (nine point) more accurate to the exact solution.

Example 4.2: A 2D Poisson equation with homogeneous Dirichlet boundary conditions DBCs is shown below.

$$-\left(fxx + fyy\right) = 2\pi^2 \sin\left(\pi x\right)\sin\left(\pi\, y\right),\ \ 0 < x < 2,\ \ 0 < y < 4,$$

$$f\left(x,0\right) = 0,\ f\left(x,4\right) = 0,$$

$$f\left(0,y\right) = 0,\ f\left(2,y\right) = 0.$$

The exact solution of the above problem on the region [02]×[04] is $f_{exact}(x,y) = sin(\pi x)sin(\pi y)$. The problem is solved by applying fourth-order CFDS and six-order CFDS methods and assuming M = N.

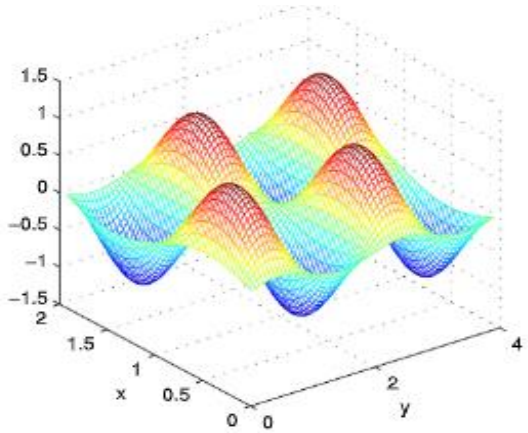

Fig.5. Plot of Solution the proposed fourth-order method

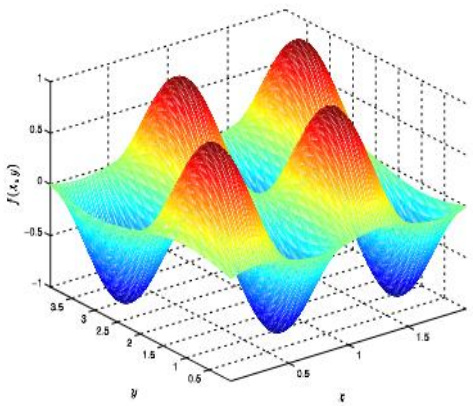

Fig.6. Plot of Solution the proposed sixth-order method

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Table 5. Comparison of error between the proposed sixth-order CFDS and a current fourth-order CFDS on Example 4.2

M=N	Error	
	Fourth-order	**Six-order**
32	8.77e-06	2.47e-08
64	5.48e-07	3.86e-10
128	3.42e-08	6.04e-12
256	2.14e-09	4.48e-13
512	1.34e-10	4.40e-13
1024	8.30e-12	4.39e-13

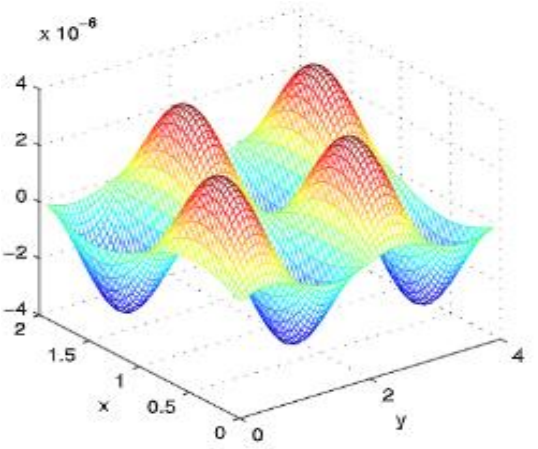

Fig.7. Plot of Error the proposed fourth-order method

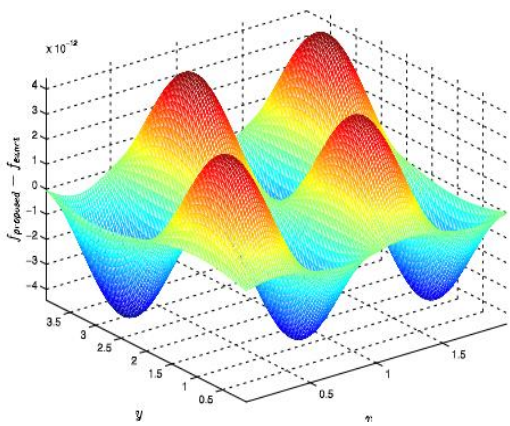

Fig.8. Plot of Error the proposed sixth-order method

The number of grid points is increased in this example as well, which results in a decrease in error, ensuring the stability of the six-order technique. Table 5 further demonstrates that the six-order strategy outperforms a fourth-order scheme currently in use. [26]

V. Conclusion

With the use of Taylor series expansions, we presented forward, backward, and centered difference approximations of first and higher derivatives in this study. Then, using constrained grid sizes, we computed more precise solutions using high-order compact finite difference techniques. In order to derive finite difference approximations for high-order derivatives in the truncation error, the high-order compact method operates on differential equations as an auxiliary relation. Finally, by solving the two-dimensional Poisson equation in a rectangular domain and contrasting the outcomes of the conventional technique with fourth order compact finite difference, we demonstrated the effectiveness of CFD.

References

[1] Pramanik, S., & Anwar, S. "Electrochemical model-based charge optimization for lithium-ion batteries." *Journal of Power Sources*, 313, 164-177 (2016).

[2] Mickens, R. E. "Nonstandard finite difference schemes for differential equations." *Journal of Difference Equations and Applications*, 8(9), 823-847 (2002).

[3] Sun, H., & Zhang, J. "A high order finite difference discretization strategy based on extrapolation for convection diffusion equations." *Numerical Methods for Partial Differential Equations*, 20(1), 18-32 (2004).

[4] Ye, L., Wu, X., Duan, H. Y., Geng, X., Chen, B., Gu, Y. Q., ... & Zhang, J. "The in vitro and in vivo biocompatibility evaluation of heparin–poly (ε-caprolactone) conjugate for vascular tissue engineering scaffolds." *Journal of Biomedical Materials Research Part A*,100(12), 3251-3258 (2012).

[5] El Morsy, S. A. "Logarithmic Finite Difference Method Applied to KdVB Equation." *American Academic & Scholarly Research Journal*, 4(2), 41-48 (2012).

[6] Izadian, J., Ranjbar, N., & Jalili, M. "The generalized finite difference method for solving elliptic equation on irregular mesh." *World Applied Sciences Journal*, (ISSN, 1818, 4952 (2013).

[7] Zhai, S., Feng, X., & He, Y. "A new method to deduce high-order compact difference schemes for two-dimensional Poisson equation.", *Applied Mathematics and Computation*, 230, 9–26 (2014).

[8] Zapata, M. U., & Balam, R. I. "High-order implicit finite difference schemes for the two-dimensional Poisson equation." *Applied Mathematics and Computation*, 309, 222-244 (2017).

[9] Xia, H., & Gu, Y. "Generalized finite difference method for electroelastic analysis of three-dimensional piezoelectric structures." *Applied Mathematics Letters*, 117, 107084 (2021).

[10] Jin, S., Liu, N., & Yu, Y. "Time complexity analysis of quantum difference methods for linear high dimensional and multiscale partial differential equations." *arxiv preprint arXiv:2202.04537*(2022).

[11] Lu, J.F., Guan, Z. "Numerical Methods for Partial Differential Equations." *Qinghua University Press, Beijing* (2003).

[12] Sun, Z.Z. "Numerical Methods for Partial Differential Equations." Science Press, Beijing (2005).

[13] Zhang, J. "An explicit fourth-order compact finite difference scheme for three-dimensional convection–diffusion equation." *Communications in numerical methods in engineering*. 14(3), 209–218 (2010).

[14] Gurarslan, G., Karahan, H., Alkaya, D., et al. "Numerical solution of advection–diffusion equation using a sixth-order compact finite difference method." *Mathematical Problems in Engineering*. (2013).

[15] Singer, I., Turkel, E. "High-order finite difference methods for the Helmholtz equation. Comput." *Computer methods in applied mechanics and engineering*, 163(1–4), 343–358 (1998).

[16] Duressa, G. F., Bullo, T. A., & Kiltu, G. G. "Fourth order compact finite difference method for solving one dimensional wave equation." *International Journal of Engineering and Applied Sciences*, *8*(4), 30-39 (2016).

[17] Li, L., Jiang, Z., & Yin, Z. "Fourth-order compact finite difference method for solving two-dimensional convection–diffusion equation". *Advances in Difference Equations*, *2018*(1), 1-24(2018).

[18] Sari, M., Gürarslan, G., Zeytinoglu, A. "High-order finite difference schemes for solving the advection–diffusion ˇ equation." *Mathematical and Computational Applications*.15(15), 449–460 (2010).

[19] Gurarslan, G. "Accurate simulation of contaminant transport using high-order compact finite difference schemes." *Journal of Applied Mathematics*, (2014).

[20] Cui, M.R. "Compact finite difference method for the fractional diffusion equation." *Journal of Computational Physics*. 228(20), 7792–7804 (2009).

[21] Li, Y.B., Kim, J. "An efficient and stable compact fourth-order finite difference scheme for the phase field crystal equation." *Computer Methods in Applied Mechanics and Engineering*. 319, 194–216 (2017).

[22] Chapra, S. C., & Canale, R. P. "Numerical methods for engineers" (Vol. 1221). New York: Mcgraw-hill. (2011).
[23] Romao, E. C., Aguilar, J. C. Z., de Campos, M. D., & de Moura, L. F. M.: "Central Difference Method Of $o(\Delta x^6)$ In Solution Of The Cdr Equation With Variable Coefficients And Robin Condition." *International Journal of Applied Mathematics*, 25(1), 139-153 (2012).
[24] De A.K., Eswaran V. "Analysis of a new high resolution upwind compact scheme". *Journal of computational physics*. 218, 398-416 (2006).
[25] Lele S. K, "Compact finite difference scheme with spectral-like resolution." *Journal of computational physics*, 103, 16-42 (1992).
[26] Gatiso, A. H., Belachew, M. T., & Wolle, G. A. "Sixth-order compact finite difference scheme with discrete sine transform for solving Poisson equations with Dirichlet boundary conditions." *Results in Applied Mathematics*, 10, 100-148 (2021).
[27] Kahlaf, S. J., & Mhassin, A. A. "Numerical solution of a two-dimensional Laplace equation with Dirichlet boundary conditions." *Journal of Physics: Conference Series*, 1818 (1), 012193 (2021).

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