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An Improved Order Seven Hybrid-Method for the Integration of Stiff First-Order Differential Equations

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Abstract— *This research deals with the derivation of an improved two-third step hybrid-block approach for the numerical integration of stiff first-order ordinary differential equations with initial values. The method derived from the collocation and interpolation of the basis function (power series) gives rise to a continuous implicit method. The estimation approach at the off-grid points gives rise to the continuous implicit linear multistep method. The evaluation of the continuous method at various points yields the block method. We investigated the following basic features: the order and error constant, zero stability, consistency, and convergence of the block method to test its efficiency. The new block method was used to solve some stiff problems to generate more efficient results when matched with some existing authors solving the same stiff first-order problems. Thus, the method can be used as an effective tool to solve first-order stiff problems.*

Keywords— *Differential Equations, Stiff problems, Implicit Hybrid Method, Interpolation and Collocation.*

I. INTRODUCTION

Differential equations are important mathematical tools used to produce models for the physical phenomenon. Most often, differential equations may not have an analytical solution. A numerical method is a good tool that assists to proffer solutions to problems that are impossible or very difficult to solve in analytical ways (Hurol, 2013). In this work, we consider a First-order Ordinary Differential Equation with initial values is of the form

$$y' = f(x, y) \quad y(x_0) = 0, \quad x \in [a, b] \tag{1}$$

The solution is in the range where and is finite and is a continuously differentiable function within an interval satisfying (1). Many authors in the literature have developed numerical methods such as Euler's Method, Runge-Kutta method, and Predictor-Corrector methods to solve the problem (1) to serve as solutions to differential equations. Other approaches recently used for solving the above problem consist of the Block formulation and Taylor series technique (Koroche, 2021; Nurujjaman, 2020; Ukpebor & Omole, 2020; Olanegan & Aladesote, 2020; Raymond et al., 2018; Ramos, 2017; Abdulaziz et al., 2017; Olanegan et al., 2015; Hurol, 2013).

Euler's method is the simplest and the most fundamental numerical method for solving first-order initial value problem. It is one of the earliest and basic methods. It provides the approximation for the solution of a differential equation (Hurol, 2013). Also, Runge-Kutta and multistep methods are well-known methods that have being used largely for solving (1). The Runge-Kutta method is self-starting and easy to program but may require subroutines to handle systems of ordinary differential equations (Ramos, 2017).

Thus, the predictor-corrector methods have greater accuracy and the error-estimating ability. However, when predictor-corrector methods are used, Runge-Kutta methods still find application in starting the computation with interval changing in the interval of integration. Despite the breakthroughs in the methods, these methods still require lots of time to write subroutines, slow rate of convergence, instability, low accuracy (Koroche, 2021). Hence, the need for the formulation of the Block method, the block formulation has the benefit of being self- starting, more efficient in terms of cost implementation, time of execution and accuracy, and were developed to take care of the noticed difficulties of predictor-corrector methods (Ramos, 2017).

Collocation is a projection method for solving integral and differential equations in which the approximate solution is determined from the condition that the equation must be stratified at given points. It involves the determination of an approximate solution in a set of functions called the basis function. Therefore, our interest in this paper is to present an efficient block method through collocation that retains the traditional advantage of Runge-Kutta of being self-starting and efficient in implementation.

II. METHOD

Consider the power series approximation as follows:

$$y = \sum_{n=0}^{k} a_n x^n \tag{2}$$

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where $k = \dfrac{2}{3}$

The first derivative of (2) produces:

$$y' = \sum_{n=0}^{k} n a_n x^{n-1} \tag{3}$$

Interpolating (2) at $x = x_{n+\tau}, \tau = \dfrac{1}{3}$ and collocating (3) at $x = x_{n+\varsigma}, \varsigma = 0, \dfrac{1}{9}, \dfrac{2}{9}, \dfrac{1}{3}, \dfrac{4}{9}, \dfrac{5}{9}, \dfrac{2}{3}$ respectively.

$$\begin{bmatrix} 1 & x_{n+\frac{1}{3}} & x^2_{n+\frac{1}{3}} & x^3_{n+\frac{1}{3}} & x^4_{n+\frac{1}{3}} & x^5_{n+\frac{1}{3}} & x^6_{n+\frac{1}{3}} & x^7_{n+\frac{1}{3}} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+\frac{1}{9}} & 3x^2_{n+\frac{1}{9}} & 4x^3_{n+\frac{1}{9}} & 5x^4_{n+\frac{1}{9}} & 6x^5_{n+\frac{1}{9}} & 7x^6_{n+\frac{1}{9}} \\ 0 & 1 & 2x_{n+\frac{2}{9}} & 3x^2_{n+\frac{2}{9}} & 4x^3_{n+\frac{2}{9}} & 5x^4_{n+\frac{2}{9}} & 6x^5_{n+\frac{2}{9}} & 7x^6_{n+\frac{2}{9}} \\ 0 & 1 & 2x_{n+\frac{1}{3}} & 3x^2_{n+\frac{1}{3}} & 4x^3_{n+\frac{1}{3}} & 5x^4_{n+\frac{1}{3}} & 6x^5_{n+\frac{1}{3}} & 7x^6_{n+\frac{1}{3}} \\ 0 & 1 & 2x_{n+\frac{4}{9}} & 3x^2_{n+\frac{4}{9}} & 4x^3_{n+\frac{4}{9}} & 5x^4_{n+\frac{4}{9}} & 6x^5_{n+\frac{4}{9}} & 7x^6_{n+\frac{4}{9}} \\ 0 & 1 & 2x_{n+\frac{5}{9}} & 3x^2_{n+\frac{5}{9}} & 4x^3_{n+\frac{5}{9}} & 5x^4_{n+\frac{5}{9}} & 6x^5_{n+\frac{5}{9}} & 7x^6_{n+\frac{5}{9}} \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x^2_{n+\frac{2}{3}} & 4x^3_{n+\frac{2}{3}} & 5x^5_{n+\frac{2}{3}} & 6x^5_{n+\frac{2}{3}} & 7x^6_{n+\frac{2}{3}} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{3}} \\ f_n \\ f_n + \frac{1}{9} \\ f_{n+\frac{2}{9}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{4}{9}} \\ f_{n+\frac{5}{9}} \\ f_{n+\frac{2}{3}} \end{bmatrix} \tag{4}$$

Finding the unknown $a_j, j = 0(1)7$ in equation (4) via Gaussian elimination procedures which are then filled into equation (2) to produce implicit continuous scheme in equation (5)

$$y(t) = \alpha_{\frac{1}{3}}(t)\, y_{n+\frac{1}{3}} + h\left[\sum_{v=0}^{\kappa} \beta_v(t)\, f_{n+v} + \beta_0(t)\, f_n\right] \tag{5}$$

for $v = \dfrac{1}{9}, \dfrac{2}{9}, \dfrac{1}{3}, \dfrac{4}{9}, \dfrac{5}{9}, \dfrac{2}{3}$ as the hybrid points.

Now, using the transformation

$$t = \frac{x - x_n}{h}, \; \frac{dt}{dx} = \frac{1}{h} \tag{6}$$

The coefficients of y_{n+j} and f_{n+j} are obtained in terms of t to be the continuous method for equation (6) as follows:

$$\alpha_{\frac{1}{3}} = 8t$$

$$\beta_0 = h\left[\frac{59049}{560}t^7 - \frac{45927}{160}t^6 + \frac{5103}{16}t^5 - \frac{11907}{64}t^4 + \frac{609}{10}t^3 - \frac{441}{40}t^2 + t - \frac{137}{4032}\right]$$

$$\beta_{\frac{1}{9}} = h\left[-\frac{117147}{280}t^7 + \frac{6561}{4}t^6 - \frac{67797}{40}t^5 + \frac{7047}{8}t^4 - \frac{2349}{10}t^3 + 27t^2 - \frac{9}{56}\right]$$

$$\beta_{\frac{2}{9}} = h\left[\frac{177147}{112}t^7 - \frac{124659}{32}t^6 + \frac{299619}{80}t^5 - \frac{112023}{64}t^4 + \frac{3159}{8}t^3 + \frac{135}{4}t^2 - \frac{129}{2240}\right]$$

$$\beta_{\frac{1}{3}} = h\left[-\frac{59049}{28}t^7 + \frac{19683}{4}t^6 - \frac{88209}{20}t^5 + \frac{7533}{4}t^4 + 381t^3 + 30t^2 - \frac{34}{315}\right]$$

$$\beta_{\frac{4}{9}} = h\left[\frac{177147}{112}t^7 + \frac{111537}{32}t^6 + \frac{234009}{80}t^5 - \frac{74601}{64}t^4 + \frac{891}{4}t^3 - \frac{135}{8}t^2 + \frac{81}{2240}\right]$$

$$\beta_{\frac{5}{9}} = h\left[-\frac{177147}{280}t^7 + \frac{6561}{5}t^6 - \frac{41553}{40}t^5 + \frac{3159}{8}t^4 - \frac{729}{10}t^3 + \frac{27}{5}t^2 - \frac{3}{280}\right]$$

$$\beta_{\frac{2}{3}} = h\left[\frac{59049}{560}t^7 - \frac{6561}{32}t^6 + \frac{12393}{80}t^5 - \frac{3645}{64}t^4 + \frac{411}{40}t^3 - \frac{3}{4}t^2 - \frac{29}{20160}\right] \tag{7}$$

Evaluating the continuous scheme at the non-interpolation points produces the discrete scheme as follows:

$$y_{n+\frac{2}{3}} - y_{n+\frac{1}{3}} = \frac{h}{20160}\left[685f_{n+\frac{2}{3}} + 3240 f_{n+\frac{5}{9}} + 1161 f_{n+\frac{4}{9}} + 2176 f_{n+\frac{1}{3}} - 729 f_{n+\frac{2}{9}} + 216 f_{n+\frac{1}{9}} - 29 f_n\right] \tag{8}$$

$$y_{n+\frac{5}{9}} - y_{n+\frac{1}{3}} = \frac{h}{34020}\left[-37 f_{n+\frac{2}{3}} + 1398 f_{n+\frac{5}{9}} + 4863 f_{n+\frac{4}{9}} + 1328 f_{n+\frac{1}{3}} + 33 f_{n+\frac{2}{9}} - 30 f_{n+\frac{1}{9}} + 5 f_n\right] \tag{9}$$

$$y_{n+\frac{4}{9}} - y_{n+\frac{1}{3}} = \frac{h}{544320}\left[271 f_{n+\frac{2}{3}} - 2760 f_{n+\frac{5}{9}} + 30819 f_{n+\frac{4}{9}} + 37504 f_{n+\frac{1}{3}} - 6771 f_{n+\frac{2}{9}} + 1608 f_{n+\frac{1}{9}} - 199 f_n\right] \tag{10}$$

$$y_{n+\frac{2}{9}} - y_{n+\frac{1}{3}} = \frac{h}{544320}\left[191 f_{n+\frac{2}{3}} - 1608 f_{n+\frac{5}{9}} + 6771 f_{n+\frac{4}{9}} - 37504 f_{n+\frac{1}{3}} - 30819 f_{n+\frac{2}{9}} + 2760 f_{n+\frac{1}{9}} - 271 f_n\right] \tag{11}$$

$$y_{n+\frac{1}{9}} - y_{n+\frac{1}{3}} = \frac{h}{34020}\left[-5 f_{n+\frac{2}{3}} + 30 f_{n+\frac{5}{9}} - 33 f_{n+\frac{4}{9}} - 1328 f_{n+\frac{1}{3}} - 4863 f_{n+\frac{2}{9}} - 1398 f_{n+\frac{1}{9}} + 37 f_n\right] \tag{12}$$

$$y_n - y_{n+\frac{1}{3}} = \frac{h}{20160}\left[29 f_{n+\frac{2}{3}} - 216 f_{n+\frac{5}{9}} + 729 f_{n+\frac{4}{9}} - 2176 f_{n+\frac{1}{3}} - 1161 f_{n+\frac{2}{9}} - 3240 f_{n+\frac{1}{9}} - 685 f_n\right] \tag{13}$$

Combining the discrete schemes in matrix form, by using matrix inversion, the methods produce a block method as follow:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{2}{3}} \\ y_{n+\frac{5}{9}} \\ y_{n+\frac{4}{9}} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{9}} \\ y_{n+\frac{1}{9}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{2}{3}} \\ y_{n-\frac{5}{9}} \\ y_{n-\frac{4}{9}} \\ y_{n-\frac{1}{3}} \\ y_{n-\frac{2}{9}} \\ y_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{41}{1260} \\ 0 & 0 & 0 & 0 & 0 & \frac{3715}{108864} \\ 0 & 0 & 0 & 0 & 0 & \frac{286}{8505} \\ 0 & 0 & 0 & 0 & 0 & \frac{137}{4032} \\ 0 & 0 & 0 & 0 & 0 & \frac{1139}{34020} \\ 0 & 0 & 0 & 0 & 0 & \frac{19087}{544320} \end{bmatrix} \begin{bmatrix} f_{n-\frac{2}{3}} \\ f_{n-\frac{5}{9}} \\ f_{n-\frac{4}{9}} \\ f_{n-\frac{1}{3}} \\ f_{n-\frac{2}{9}} \\ f_n \end{bmatrix}$$

$$+h \begin{bmatrix} \frac{41}{1260} & \frac{6}{35} & \frac{3}{140} & \frac{68}{315} & \frac{3}{140} & \frac{6}{35} \\ -\frac{275}{108864} & \frac{235}{4536} & \frac{3875}{36288} & \frac{250}{1701} & \frac{2125}{36288} & \frac{725}{4536} \\ -\frac{8}{8505} & \frac{16}{2835} & \frac{58}{2835} & \frac{1504}{8505} & \frac{128}{2835} & \frac{464}{2835} \\ -\frac{29}{20160} & \frac{3}{280} & -\frac{81}{2240} & \frac{34}{315} & \frac{129}{2240} & \frac{9}{56} \\ -\frac{37}{34020} & \frac{22}{2835} & -\frac{269}{11340} & \frac{332}{8505} & \frac{11}{11340} & \frac{94}{567} \\ -\frac{863}{544320} & \frac{263}{22680} & -\frac{6737}{181440} & \frac{586}{8505} & -\frac{15487}{181440} & \frac{2713}{22680} \end{bmatrix} \begin{bmatrix} f_{n+\frac{2}{3}} \\ f_{n+\frac{5}{9}} \\ f_{n+\frac{4}{9}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{9}} \\ f_{n+\frac{1}{9}} \end{bmatrix} \tag{14}$$

Basic Properties of the Method

The analysis of the basic properties of the block are investigated as follows:

Order and Local Truncation Error

We define the Local Truncation Error (LTE) with equation (14) above as:

$$L\left[y(x):h\right] = \sum_{j=0}^{k}\left[\alpha_j y\left(x_n + jh\right)\right]$$

where, $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$. This is expanded by Taylor series point x to obtain

$$L\left[y(x):h\right] = C_0 y(x) + C_1 h y'(x) + \cdots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + \cdots$$

where $C_0, C_1, C_2, C_p, \cdots, C_{p+1}$ are obtained as

$$C_0 = \sum_{j=0}^{k}\alpha_j, C_1 = \sum_{j=0}^{k} j\alpha_j, C_2 = \frac{1}{2!}\sum_{j=1}^{k} j^2\alpha_j \cdots C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q\alpha_j - q(q-1)\sum_{j=1}^{k}\beta_j j^{q-2}\right]$$

where $C_{p+1}h^{p+1}y^{p+1}(x_n) + 0\left(h^{p+1}\right)$ is the Local Truncation Error (LTE) at the point x_n. Hence, equation (14) is of uniform order $\begin{bmatrix}7 & 7 & 7 & 7 & 7\end{bmatrix}^T$ and Local Truncation Error of

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$$\left[-\frac{1}{4285540224} \quad \frac{1}{3654044313} \quad \frac{1}{5113400743} \quad -\frac{1}{9346110874} \quad \frac{1}{6784214000}\right]^T$$

Zero Stability of the Block

Definition: A Block method is said to be zero-stable, if no root of the first characteristics polynomials $\rho(r)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than two. Thus, equation (14) is zero stable since no root of the first characteristics polynomial $\rho(r)$ has modulus greater than one that is $r \le 1$ (Olanegan et al., 2015).

Consistency

A block method is said to be consistent if it has order $p \ge 1$. Therefore, the block is consistent, since $p = 7$

Convergence of the Method

A linear multistep method is said to be convergent if it is consistent and zero stable. Thus, the method converges since it satisfies this condition, (Olanegan et al., 2015).

Numerical Experiments and Results

This section presented some experiments carried out with the new method to solve some stiff first-order differential equations. The computational results presented in the table are compared with some existing methods in the literature.

Experiment 1

Consider the stiff first-order initial value problem:

$$y' = x - y \qquad y(0) = 1, \qquad h = 0.1$$

Theoretical Solution: $y(x) = e^{-\lambda x}$

(Ukpebor and Omole 2020).

Experiment 2

$$y' = -y, \qquad y(0) = 1, \qquad h = 0.1$$

Theoretical Solution: $y(x) = e^{-x}$

(Ukpebor & Omole, 2020; Raymond et al., 2018)

Experiment 3

Consider the highly stiff first-order initial value problem:

$$y = -\lambda y \qquad y(0) = 1, \lambda = 10, h = 0.1$$

Theoretical Solution: $y(x) = e^{-\lambda x}$

(Ukpebor and Omole 2020; and Raymond, et al., 2018)

Numerical Results and Comparison

Note: The following notations are used for representation in the tables below:
EMUG - Error in (Ugbebor & Omole, 2020)
EMRD - Error in (Raymond et al., 2018)

TABLE 1: Results and Comparison for Experiment 1

x	Exact Solution	Numerical Solution	Error	EMUG
0.1	1.009896156377111400	1.009896156376720800	3.905765E-13	3.00E-11
0.2	1.010156599225359200	1.010156599224959500	3.996803E-13	2.00E-11
0.3	1.010417043451421000	1.010417043450990500	4.305445E-13	1.90E-10
0.4	1.010677489090640300	1.010677489090141500	4.987122E-13	1.30E-10
0.5	1.010937936178362100	1.010937936177746200	6.159517E-13	1.00E-10
0.6	1.011198384749932600	1.011198384749315600	6.170620E-13	2.00E-10
0.7	1.011458834840698500	1.011458834840079300	6.192824E-13	2.00E-10
0.8	1.011719286486008700	1.011719286485377000	6.317169E-13	2.00E-10
0.9	1.011979739721212900	1.011979739720540100	6.727952E-13	2.00E-10
1.0	1.012240194581662100	1.012240194580898500	7.636114E-13	3.00E-10

TABLE 2: Results and Comparison for Experiment 2

x	Exact Solution	Numerical Solution	Error	EMUG	EMRD
0.1	-0.105170918075647710	-0.105170917141109390	9.345383E-10	1.00E-10	1.95E-11
0.2	-0.110710610355705170	-0.110710609258201450	1.097504E-09	0.00000	2.47E-09
0.3	-0.107937301909804660	-0.107937300893886810	1.015918E-09	0.00000	4.18E-08
0.4	-0.106553245497890580	-0.106553244522687970	9.752026E-10	0.00000	3.09E-07
0.5	-0.109323089473978510	-0.109323088417293890	1.056685E-09	1.00E-10	1.45E-06
0.6	-0.112099866722986660	-0.112099865584609090	1.138378E-09	1.00E-10	1.31E-06
0.7	-0.117674290502672860	-0.117674289178823550	1.323849E-09	0.00000	1.19E-06
0.8	-0.114883594599782150	-0.114883593368785900	1.230996E-09	1.00E-10	1.10E-06
0.9	-0.113490860746536140	-0.113490859561878250	1.184658E-09	2.00E-10	1.16E-06
1.0	-0.116278070458871290	-0.116278069181478460	1.277393E-09	1.00E-10	1.76E-06

TABLE 3: Results and Comparison for Experiment 3

x	Exact Solution	Numerical Solution	Error	EMUG	EMRD
0.1	0.860371828030024540	0.860371884775198210	5.674517E-08	3.00E-11	1.95E-11
0.2	0.859028546760510660	0.859028605490282480	5.872977E-08	2.00E-11	2.47-E09
0.3	0.859699925035657950	0.859699983280735160	5.824508E-08	1.90E-10	4.18-E08
0.4	0.860035810917315290	0.860035868565216190	5.764790E-08	1.30E-10	3.09-E07
0.5	0.859364170333800530	0.859364228939572760	5.860577E-08	1.00E-10	1.45-E06
0.6	0.858693054264576540	0.858693112810855470	5.854628E-08	2.00E-10	1.31-E06
0.7	0.857352394030853730	0.857352454593685360	6.056283E-08	2.00E-10	1.19E-06
0.8	0.858022462300026540	0.858022522370661790	6.007064E-08	2.00E-10	1.10E-06
0.9	0.858357692794805780	0.858357752258595790	5.946379E-08	2.00E-10	1.16E-06
1.0	0.857687362729086520	0.857687423166135330	6.043705E-08	3.00E-10	1.76E-06

III. CONCLUSION

This research has focused on the formulation, exploration, and execution of the Two-third step improved block approach for the numerical integration of stiff first-order differential equations. The method was found to satisfy some basic properties of a numerical method. The results are shown in Tables 1 to 3. The results, when compared to some existing works as demonstrated in the tables above, performed favorably. Therefore, the method is said to be trustworthy and proficient to solve the stiff problem of first-order Ordinary Differential Equations.

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