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On the Algebrian Equation of the Third Degree

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Abstract— *This article is the continuation of a work already published in [1]. It presents a new approach to solving third degree equations with real coefficients using the calculation of areas.*

Keywords— *Equation, method, coefficients, reals and areas.*

I. Introduction

Solving the third and fourth degree equations gave considerable boost to algebra in the centuries that followed. However, despite all the efforts made by mathematicians, it took almost 300 years for Abel and then Galois to finally provide the (negative!) Answer to the question of the resolubility by radicals of higher degree equations. In the meantime, significant progress was made including the appearance of modern algebraic notations and the systematic use of negative, even complex, numbers. An important property was discovered (although the proof, brought by Gauss, was slow to come): any equation of degree n admits exactly n solutions in the set of complex numbers [2].

As the number 50 of Tangente (June 1996, page 27) reminded us, the CARDAN formula was published for the first time in 1545. It has two cubic roots and two square roots, to be calculated by hand. . , which is long and delicate.

This is why we imagined simple and fast graphical methods, allowing to solve the equations of the third and the fourth degree, which was the second method of solving the equations. Among the easiest are those that use the parable $y = x^2$ that we draw once and for all. This parabola is cut by a circumference whose radius and coordinates of the center must be calculated [3]. It then suffices to read the abscissas of the points of intersection of the parabola and the circumference: these are the roots sought. This method is very easy and quick. Concerning our research, we used the area method.

II. Preliminary

A. Fitness

We will consider in this work the polynomial function of degree three defined by:

$$f(X) = aX^3 + bX^2 + cX + d \qquad (1)$$

where a, b, c, d are reals with $a \neq 0$, and the equation

$$f(X) = 0 \qquad (2)$$

We will first assume that $X \in \mathbb{R}$ and then we will examine the case $X \in \mathbb{C}$

Proposition 1 :

Let f be the polynomial function defined by (1). So

$$b^2 = 3ac \Leftrightarrow \exists\ (m \neq 0, p, q \in \mathbb{R}), aX^3 + bX^2 + cX + d = (mX + p)^3 - q^3$$

In this case the roots of equation (2) are: $X_1 = \dfrac{q-p}{m}$; $X_2 = \dfrac{-2p - q + iq\sqrt{3}}{2m}$; $X_3 = -\dfrac{2p + q + iq\sqrt{3}}{2m}$

Where $i^2 = -1, m = \sqrt[3]{a}, p = \dfrac{b^2}{3m^2}, q = \sqrt[3]{p^3 - d}$

In particular equation (2) admits a triple root if and only if q=0.

Evidence. Assuming that $aX^3 + bX^2 + cX + d = (mX + p)^3 - q^3$, where $m \neq 0$ p and q are reals and by comparing the coefficients we get $b^2 = 3ac$.

Conversely, suppose that $b^2 = 3ac$. So we have $b = \pm\sqrt{3ac}$ et $f(X) = aX^3 \pm \sqrt{3ac}X^2 + cX + d$.

Four cases arise:

1st case $c = d = 0$, we obtain $m = \sqrt[3]{a}$, $p = q = 0$.

2nd case : $c = 0$ et $d \neq 0$. We obtain $m = \sqrt[3]{a}$, $p = 0$ et $q = -\sqrt[3]{d}$.

3rd case : $c \neq 0$ et $d = 0$. We obtain $m = \sqrt[3]{a}$ et $p = q = \dfrac{\sqrt{3ac}}{3m^2}$.

4th case : $c \neq 0$ et $d \neq 0$. We find $m = \sqrt[3]{a}$, $p = \dfrac{\sqrt{3ac}}{3m^2}$, $q = \sqrt[3]{p^3 - d}$.

Equation (2) is then written:

$$(mX+p)^3 - q^3 = (mX + p - q)\left[(mX+p)^2 + q(mX+p) + q^2\right] \qquad (3)$$

Hence the roots: $X_1 = \dfrac{q-p}{m}$; $X_2 = \dfrac{-2p - q + iq\sqrt{3}}{2m}$; $X_3 = -\dfrac{2p + q + iq\sqrt{3}}{2m}$

Corollary. If $b^2 = 3ac$, then equation (2) admits at least one real root.

Proposition 2 :

Let f be the polynomial function defined by (1). If $b^2 = 3ac$ and if the roots are simple, then equation (2) admits only one real root. See in [4] for **proof**.

B. Use of Areas

In this section we will assume that a > 0 and that the roots of equation (2) are real. Then, we will denote by A_1 and A_2 the areas of the surfaces determined by the curve representative of the function f, the x-axis and the straight lines of equations $X = X_1, X = X_2$ and $X = X_3$. (See fig. 1). , we will assume that the following conditions are satisfied:

$$\begin{cases} x_1, x_2, x_3 \in \mathbb{R}, x_1 \leq x_2 \leq x_3 , \\ r = \dfrac{1}{x_3 - x_1} , x_1 \neq x_3, \\ \mathcal{A}_1 - \mathcal{A}_2 = k , \ k \in \mathbb{R}, \\ \mathcal{A}_1 + \mathcal{A}_2 = l , \ l \in \mathbb{R}_+^\star . \end{cases} \quad (4)$$

The areas A_1 and A_2 are given by the relations:

$$A_1 = \frac{a}{4}\left(X_2^4 - X_1^4\right) + \frac{b}{3}\left(X_2^3 - X_1^3\right) + \frac{c}{2}\left(X_2^2 - X_1^2\right) + d\left(X_2 - X_1\right) \quad (5)$$

$$A_2 = -\frac{a}{4}\left(X_3^4 - X_2^4\right) - \frac{b}{3}\left(X_3^3 - X_2^3\right) - \frac{c}{2}\left(X_3^2 - X_2^2\right) - d\left(X_3 - X_2\right) \quad (6)$$

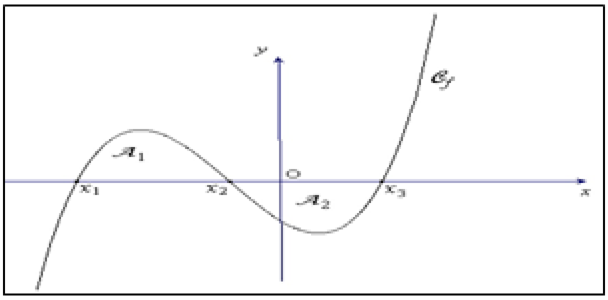

Fig. 1. Representative curve of the function f

The case $k = 0$ is examined in [1].

For $k \neq 0$, we obtain the following result:

Theorem 1. If $k \neq 0$ and if conditions (3) are satisfied, then equation (2) admits the following real roots*:*

$$X_1 = -\frac{3a + 2br + 12kr^4}{6ar} , \qquad X_2 = \frac{12kr^3 - b}{3a} , \qquad X_3 = \frac{3a - 2br - 12kr^4}{6ar}$$

Proof. **The relationship** $A_1 - A_2 = k \ \left(k \in \mathbb{R}\right)$ **leads to the equation :**

$$3a^3X_2^3 + 3a^2bX_2^2 + ab^2X_2 + b^3 - 4abc + 12a^2d - 12a^2kr = 0 \quad (7)$$

According to propositions 1 and 2, equation (7) admits the real root $X_2 = \dfrac{u - b}{3a}$ or

$$u = \sqrt[3]{-8b^3 + 36abc - 108a^2d + 108a^2kr} \quad (8)$$

X_2 will be solution of equation (2) if and only if:

$$u^3 - 3\left(b^2 - 3ac\right)u + 2b^3 - 9abc + 27a^2d = 0 \quad (9)$$

The relation (9) can still be written:

$$u^3 - 4\left(b^2 - 3ac\right)u + 36a^2kr = 0 \quad (10)$$

Express u as a function of k and r. To do this, let's put relation (10) in the form:

$$4\left(b^2 - 3ac\right)u - u^3 = 36a^2kr$$

and let us raise the two members of this equation squared. We obtain :

$$u^6 - 8\left(b^2 - 3ac\right)u^4 + 16\left(b^2 - 3ac\right)^2 u^2 = 1296a^4k^2r^2 \quad (11)$$

We show that $r^2 = \dfrac{3a^2}{-u^2 + 4\left(b^2 - 3ac\right)}$

Relation (11) is then written:

$$u^6 - 8\left(b^2 - 3ac\right)u^4 + 16\left(b^2 - 3ac\right)^2 u^2 = 1296a^4k^2 \frac{3a^2}{-u^2 + 4\left(b^2 - 3ac\right)}$$

or :

$$u^2\left[4\left(b^2 - 3ac\right) - u^2\right]^3 = 3888a^6k^2 \quad (12)$$

Taking into account the relation (9) we finally obtain

$$u = 12kr^3 \quad (13)$$

Otherwise using the condition $X_2 - X_1 = \dfrac{1}{r}$ and taking into account that

$X_1 + X_2 + X_3 = -\dfrac{b}{a}$, we then obtain the real roots of equation (2).

Extreme cases $X_1 = X_2 \neq X_3$ et $X_1 \neq X_2 = X_3$ lead to the following theorem:

Theorem 2 : Let f be the polynomial function defined by (1). If X_1, X_2 et X_3 are the real roots of f and if the discriminant $D = b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2$ of equation (2) zero, so equation (2) admits three real roots of which at least two coincide [5]. These roots are found among the numbers $\dfrac{\pm 2\sqrt{b^2 - 3ac} - b}{3a}$ et $-\dfrac{b \pm \sqrt{b^2 - 3ac}}{3a}$.

Proof. See [1].

In what follows we will ask $\alpha = b^2 - 3ac$.

Theorem 3 : If $k \neq 0$ and if conditions (3) are satisfied, then the numbers r and k of Theorem 1 can be expressed as a function of I by the following relations :

$$r = \frac{a\sqrt{3}}{\sqrt{3\alpha + \sqrt{3\alpha^2 - 36a^3\mathrm{I}}}} , \quad k = \pm\frac{1}{6a^3}\sqrt{\frac{\alpha^4 - 6a^2D\alpha - 36a^6\mathrm{I}^2}{3}}$$

, where I, particular solution of the equation $\mathrm{I}^3 + p\mathrm{I} + q = 0$, is given by the dial formula :

$$\mathrm{I} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \text{avec}$$

$$p = \frac{\alpha\left(6a^2\mathrm{D} - \alpha^3\right)}{48a^6} , \quad q = \frac{2\alpha^6 - 18a^2\mathrm{D}\alpha^3 + 27a^4\mathrm{D}^2}{1728a^9}$$

Proof. Using the relationship $A_1 + A_2 = \mathrm{I}$ and Theorem 2, we get the square equation :

$$2\left[\alpha^2 + 6a^3\mathrm{I}\right]r^4 - 6a^2\alpha r^2 - a^4 = 0 \quad (14)$$

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which allows to find the only value of r which is suitable : $r = \frac{a\sqrt{3}}{\sqrt{3\alpha + \sqrt{3\alpha^2 - 36a^3 \mathrm{I}}}}$

Relations (8) and (10) lead to the following equation:

$$\mathrm{D}r^6 - \alpha^2 r^4 + 2a^2\alpha r^2 - a^4 = 0 \quad (15)$$

By replacing r by its expression in (15) we get :

$$\left[\sqrt{\alpha^2 - 12a^3\mathrm{I}}\right]^3 = \sqrt{3}\left[3a^2\mathrm{D} + 12\mathrm{a}^3\alpha\mathrm{I} - \alpha^3\right] \quad (16)$$

On the other hand, by replacing r by its expression in (10) we get :

$$\left[\sqrt{\alpha^2 - 12a^3\mathrm{I}}\right]^3 = \sqrt{3}\left[12a^3\alpha\mathrm{I} - \frac{54\mathrm{a}^6k^2}{\alpha} - \frac{\alpha^3}{2} - \frac{18a^6\mathrm{I}^2}{\alpha}\right] \quad (17)$$

Relations (16) and (17) make it possible to express k as a function of I:

$$k = \pm\frac{1}{6a^3}\sqrt{\frac{\alpha^4 - 6a^2\mathrm{D}\alpha - 36a^6 I^2}{3}} \quad (18)$$

The relation (16) gives us the following equation :

$$1728a^9\mathrm{I}^3 + 36a^3\alpha\left[6a^2\mathrm{D} - \alpha^3\right]\mathrm{I} + 2\alpha^6 - 18a^2 D\alpha^3 + 27a^4 D^2 = 0 \quad (19)$$

a particular solution of which is given by the Cardan formula [6, 7, 8] :

$$\mathrm{I} = \sqrt[3]{-\frac{\mathrm{q}}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{\mathrm{q}}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

with

$$\mathrm{p} = \frac{\alpha\left(6a^2\mathrm{D} - \alpha^3\right)}{48a^6}, \quad \mathrm{q} = \frac{2\alpha^6 - 18a^2\mathrm{D}\alpha^3 + 27a^4\mathrm{D}^2}{1728a^9}$$

III. Conclusion

We have proposed in this modest work to approach the resolution of third degree equations with real coefficients using the area calculation. This approach allowed us to identify some interesting particular cases without resorting to the Cardan formula. In fact we have shown that if $k = 0$ we obtain simple formulas for the calculation of the roots of the third degree equation. In the general case, i.e. if $k \neq 0$, the problem becomes complex and the solutions of the equation can be obtained using Theorem 1.

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