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Higher- Order Derivatives of the Function Sine Bx and its Special Properties

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Abstract— *The study aims to find a formula in solving for the higher-order derivative of the trigonometric function f(x) = sin bx, where b ≠ 0, that is,* $f^n(x)$ *where* $n \in \mathbb{Z}^+$. *The formulas derived are* $f^{2n-1}(x) = (-1)^{n+1} b^{2n-1} \cos bx$ *and* $f^{2n}(x) = (-1)^n b^{2n} \sin bx$, *where* $b, x \in \mathbb{R}$, *for odd and even – order derivatives, respectively. Formulas were derived with the use of geometric sequence and verified through the Principle of Mathematical Induction (PMI). Special properties of high-order derivatives of the said function were determined which lead to generate some corollaries and verified by direct proof. These corollaries help us to understand the behavior of the said trigonometric function once differentiated. There are researches in Physics studying and examining the concept of trampolines and theme park roller coasters with the help high-order derivatives. They are using the term acceleration, jerk, snap or jounce, crackle and pop for 2nd, 3rd, 4th, 5th and 6th derivatives, correspondingly. Thus, the findings of this study could be of help in further studies in physics, chemistry, electronics, engineering, finance and economics, and other fields.*

Keywords— *Calculus, derivatives, higher-order derivatives, sine function, trigonometry.*

I. Introduction

According to Galileo Galilei, Mathematics is the language of Science [1]. We use Mathematics to understand a lot of occurrences. Calculus is the mathematics of motion or change. It is also used in many fields of work and study, including physics, chemistry, electronics, engineering, finance and economics [2]. Larson defined Calculus as "the branch of Mathematics that deals with limits and the differentiation and integration of functions of one or more variables." [3] There is no agreement of the names of higher order derivatives. [4]

In the study conducted by David Eager, Ann-Marie Pendrill and Nina Reistad, they used the term snap to denote the fourth derivative of displacement with respect to time. Another name for this fourth derivative is jounce. The fifth and sixth derivatives with respect to time are referred to as crackle and pop respectively. They examined third and higher order derivatives of displacement with respect to time using the trampolines and theme park roller coasters to illustrate this concept. Also, they considered the effects on the human body of different types of acceleration, jerk, snap and higher derivatives, and how they can be used in physics education to further enhance the learning and thus the understanding of classical mechanics concepts. [4]

Oronce and Mendoza defined trigonometry as "the study of triangles and the relationship among their sides and angles." [5] An example of trigonometric function that astonishes us is $f(x) = \sin x$. According to Stewart, et al, the sine and cosine functions are periodic. They also defined a periodic function as "if there is a positive number p such that $f(t+p) = f(t)$ for every t The least such positive number (if it exists) is the period of f".[6] The sine function repeats their values in any interval of length 2π. Thus, the sine function has a period of 2π.

Taking the higher-order derivative of the function $f(x) = \sin x$ gives us values or formulas that can be of use in further studies in physics, chemistry and other fields.

II. Main Results

To determine the pattern of the higher-order derivatives of the trigonometric function $f(x) = \sin bx$ where $b \neq 0$, the first few derivatives of the said function were determined. Let $f^n(x)$ be the nth - order derivative of $f(x)$, where $n \in \mathbb{Z}^+$. In this study, let $f^0(x)$ be equivalent to $f(x)$, that is, $f(x) = f^0(x) = \sin bx$.

Table 1. Some nth- order derivatives of $f(x) = \sin bx$.

nth -order derivative	n	$(2n-1)th$ - order derivative	$2nth$ - order derivative
$f(x) = \sin bx$	0	undefined.	$f^0(x) = \sin bx$
	1	$f'(x) = b\cos bx$	$f''(x) = -b^2 \sin bx$
	2	$f'''(x) = -b^3 \cos bx$	$f^{iv}(x) = b^4 \sin bx$
	3	$f^v(x) = b^5 \cos bx$	$f^6(x) = -b^6 \sin bx$
	4	$f^7(x) = -b^7 \cos bx$	$f^8(x) = b^8 \sin bx$
	5	$f^9(x) = b^9 \cos bx$	$f^{10}(x) = -b^{10} \sin bx$
	n	$f^{2n-1}(x) = ?$	$f^{2n}(x) = ?$

The pattern in the high-order derivatives of $f(x) = \sin bx$ can be observed in Table 1 which suggests that if n is odd, then the derivative of $f(x)$ is always expressed in terms of $\cos bx$ and the power of its coefficient is equal to the value of n. Similarly, if n is even, then the derivative of $f(x)$ is expressed in terms of $\sin bx$ and the power of its coefficient is equal to the value of n. Another observation is that each column forms a geometric sequence.

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A lot of patterns can be observed from the given data, $\cos bx$ is achieved whenever it is in $(2n-1)th$ derivative while $\sin bx$ is achieved whenever it is in $(2n)th$ derivative.

III. Results and Discussion

The higher-order derivatives, ($f^n(x)$), of the trigonometric function $f(x)=\sin bx$ are in geometric sequence. Suppose $b,x\in\mathbb{R}$, $n\in\mathbb{Z}^+$ and $f(x)=\sin bx$, then the nth derivative of $f(x)$ is:

Theorem 1.1

Let $f^n(x)$ be the nth-order derivative of $f(x)=\sin bx$ where $n\in\mathbb{Z}^+$. For odd or $(2n-1)th$ derivative,

$$f^{2n-1}(x)=(-1)^{n+1}b^{2n-1}\cos bx \quad (1)$$

while the even or (2n)th derivative of $f(x)$ is

$$f^{2n}(x)=(-1)^{n}b^{2n}\sin bx \quad (2)$$

Proof: Using Principle Mathematical Induction (PMI)

Let $n\in\mathbb{Z}^+$ and P_n be the statement that $f^{2n-1}(x)=(-1)^{n+1}b^{2n-1}\cos bx$. Verify whether P_1,P_2 and P_3 are true.

If $n=1$, then $f^{2n-1}(x)=f'(x)=(-1)^{1+1}b^{2(1)-1}\cos bx$

$$f'(x)=b\cos bx$$

If $n=2$, then $f^{2n-1}(x)=f'''(x)=(-1)^{3}b^{3}\cos bx$

$$f'''(x)=-b^3\cos bx$$

If $n=3$, then $f^{2n-1}(x)=f^5(x)=(-1)^{4}b^{4}\cos bx$

$$f^5(x)=b^5\cos bx$$

Referring on Table 1, P_1,P_2 and P_3 are all true. Next, we need to show that if m and k are both positive integers and P_m is true for all $3<m\le k$, then P_{k+1} is also true. If P_m is true, then $P_m=f^{2m-1}(x)=(-1)^{m+1}b^{2m-1}\cos bx$, for all $3<m\le k$.

Now, let $m=k$, we have $P_k=f^{2k-1}(x)=(-1)^{k+1}b^{2k-1}\cos bx$.

We want to show, $P_{k+1}=f^{2(k+1)-1}(x)=(-1)^{(k+1)+1}b^{2(k+1)-1}\cos bx$.

To solve for P_{k+1}, we need to solve for $f^{2k+1}(x)$ which means that we need to differentiate the P_k twice. Since $P_k=f^{2k-1}(x)$, we need to determine $f^{2k}(x)$ and $f^{2k+1}(x)$. Thus, the derivative of f^{2k-1} is f^{2k}, so

$$f^{2k-1}(x)=(-1)^{k+1}b^{2k-1}\cos bx$$
$$f^{2k}(x)=(-1)^{k+2}b^{2k}\sin bx$$

Now, take the derivative of $f^{2k}(x)$ which is the $f^{2k+1}(x)=P_{k+1}$.

$$P_{k+1}=f^{2k+1}(x)=(-1)^{k+2}b^{2k}(\cos bx)(b)$$

$$P_{k+1}=f^{2k+1}(x)=(-1)^{(k+1)+1}b^{2(k+1)-1}\cos bx$$

Thus, P_{k+1} is also true. By the Principle of Mathematical induction, P_n is true for all $n\in\mathbb{Z}^+$.

Therefore, $f^{2n-1}(x)=(-1)^{n+1}b^{2n-1}\cos bx$, where $b,x\in\mathbb{R}$ and $n\in\mathbb{Z}^+$.

Similarly, Eq. (2), $f^{2n}(x)=(-1)^{n}b^{2n}\sin bx$ can be proven by PMI.

Using (1) and (2) and the behavior of the higher-order derivatives of the function $f(x)=\sin bx$ in Table 1, corollaries can be derived. These corollaries are as follows:

Let $f(x)=\sin bx\neq 0$, where $b,x\in\mathbb{R}$ and $n\in\mathbb{Z}^+$.

Corollary 1. $f^{2n}(x)+b^{2n}f(x)=0$, n is odd. (3)

Proof:

Let $f(x)=\sin bx\neq 0$, where $b,x\in\mathbb{R}$ and $n\in\mathbb{Z}^+$ and $f^{2n}(x)=(-1)^{n}b^{2n}\sin bx$.

Suppose $f^{2n}(x)+b^{2n}f(x)=0$ and n is even. In Eq. 2 of Theorem 1.1,

$$f^{2n}(x)+b^{2n}f(x)=(-1)^{n}b^{2n}\sin bx+b^{2n}\sin bx.$$

Since n is even, then $(-1)^n=1$. Thus, since $f(x)\neq 0$, then

$$f^{2n}(x)+b^{2n}f(x)\neq 0.$$

Which is a contradiction. Hence, the assumption that n is even is false. Therefore, n must be odd.

Corollary 2. $f^{2n}(x)-b^{2n}f(x)=0$, n is even. (4)

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The proof for Corollary 2 is similar with the proof of Corollary 1.

Corollary 3. $f^{2n}(x)+b^2 f^{2n-2}(x)=0$ (5)

Proof:

Let $f(x)=\sin bx \neq 0$, where $b,x\in\mathbb{R}$ and $n\in\mathbb{Z}^+$. We need to show that $f^{2n}(x)+b^2 f^{2n-2}(x)=0$. So,

$$\begin{aligned} & f^{2n}(x)+b^2 f^{2n-2}(x) \\ &=(-1)^n b^{2n}\sin bx+b^2[(-1)^{n-1}b^{2n-2}\sin bx] \\ &=(-1)^n b^{2n}\sin bx+(-1)^{n-1}b^{2n}\sin bx \\ &=(-1)^n b^{2n}[\sin bx-\sin bx] \\ &=0 \end{aligned}$$

Thus, $f^{2n}(x)+b^2 f^{2n-2}(x)=0$.

Corollary 4. $\dfrac{f^{2n}(x)}{f^{2n-2}(x)}=-b^2$ (6)

Proof:

Let $f(x)=\sin bx \neq 0$ where $b,x\in\mathbb{R}, b\neq 0$. We need to show that $\dfrac{f^{2n}(x)}{f^{2n-2}(x)}=-b^2$.

So,

$$\frac{f^{2n}(x)}{f^{2n-2}(x)}=\frac{(-1)^n b^{2n}\sin bx}{(-1)^{n-1}b^{2n-2}\sin bx}=-b^2$$

Thus, $\dfrac{f^{2n}(x)}{f^{2n-2}(x)}=-b^2$.

The succeeding corollaries can be proven using direct proof.

Corollary 5. $\dfrac{f^{2n}(x)}{f^{2n-1}(x)}=-b\tan bx$, $f^{2n-1}(x)\neq 0$ (7)

Corollary 6. $\dfrac{f^{2n}(x)+f^{2n-1}(x)}{f''(x)+f'(x)}=(-1)^{n+1}b^{2n-2}$,

where $f''(x)+f'(x)\neq 0$ (8)

Corollary 7. $\dfrac{f^{4n}(x)+f^{4n-2}(x)}{b^{4n}-b^{4n-2}}=f(x)$ (9)

Corollary 8.

$$\sum_{n=0}^{\infty} f^n(x)=\frac{\sin bx+b\cos bx}{1+b^2}=\frac{f(x)+f'(x)}{1+b^2}$$

where $|b|<1$. (10)

Corollary 9. $\dfrac{\sum_{n=0}^{\infty} f^n(x)}{\sum_{n=0}^{\infty} f^{2n+1}(x)}=\dfrac{\tan bx}{b}=\dfrac{f(x)}{f'(x)}$, where $|b|<1$ and $f'(x)\neq 0$. (11)

Corollary 10. $\dfrac{\sum_{n=0}^{\infty} f^{2n+1}(x)}{\sum_{n=0}^{\infty} f^{2n}(x)}=b\cot bx=\dfrac{f'(x)}{f(x)}$, where $|b|<1$ and $f(x)\neq 0$. (12)

Corollary 11.

$$b\sin bx\sum_{n=0}^{\infty} f^{2n}(x)+\cos bx\sum_{n=0}^{\infty} f^{2n+1}(x)=\frac{b}{b^2+1}, n\in\mathbb{Z}^+$$

, where $|b|<1$. (13)

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